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Fractal Zeta Functions in Metric Measure Spaces

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by

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dedicated in memory of my father

Eric Bruce Henderson

a volume of margins is too narrow to contain
my gratitude for his wisdom and advice

ABSTRACT OF THE DISSERTATION

Fractal Zeta Functions in Metric Measure Spaces

by

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Dr. Michel Lapidus, Chairperson

Fractal zeta functions associated to bounded subsets of Euclidean spaces relate the geometry of a set to the spectrum of a Laplace operator defined on that set, thereby making it possible to rephrase certain spectral problems in terms of the set's geometry, and vice versa. We generalize the global theory of fractal zeta functions in Euclidean spaces to a broader class of metric spaces with finite Assouad dimension. We also introduce a local theory which gives a more refined tool for analyzing multifractal measures.

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List of Symbols and Notation

$\mathbb{N}$	the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$	3
$\dim_{\mathrm{As}}$	the measure theoretic Assouad dimension	12
$\underline{\underline{\mathfrak{M}}}^q$	the q-dimensional lower Minkowski content	13
$\overline{\overline{\mathfrak{M}}}^q$	the q-dimensional upper Minkowski content	13
$\underline{\underline{\dim}}_{\mathrm{Mi}}$	the lower Minkowski dimension	14
$\overline{\overline{\dim}}_{\mathrm{Mi}}$	the upper Minkowski dimension	14
$\dim_{\mathrm{Mi}}$	the Minkowski dimension	14
Φ	an iterated function system	16
$\mathscr{A}$	the attractor of an iterated function system	17
$\mathscr{I}$	a finite index set	17
$\mathscr{I}^*$	the collection of all finite words in $\mathscr{I}$	17
$\mathbb{i}$	the imaginary unit	18
$(\Phi, \mathfrak{p})$	a weighted self-similar iterated function system	19
$\dim_{\mathrm{loc}}\, \mu(x)$	the local dimension of μ at x	20
ζ_E	the distance zeta function	24
$D_C(\zeta_E)$	abscissa of convergence of ζ_E	30
$\tilde{\zeta}_E$	the tube zeta function	34
$D_H(\zeta_E)$	abscissa of holomorphic continuation of ζ_E	35
$\mathscr{P}_U$	the visible poles of f	39
$\mathscr{P}(f)$	the poles of f	39
$\mathscr{P}_U(\zeta_E)$	the set of visible complex dimensions of a set E	39
$\mathscr{P}(\zeta_E)$	the set of complex dimensions of a set E	39
(E, Ω)	a relative fractal drum	40
$\zeta_{E,\Omega}$	the relative distance zeta function	40
$\tilde{\zeta}_{E,\Omega}$	the relative tube zeta function	40
$\underline{\underline{\mathfrak{M}}}^q(E, \Omega)$	the lower q-dimensional Minkowski content of an RFD	40
$\overline{\overline{\mathfrak{M}}}^q(E, \Omega)$	the upper q-dimensional Minkowski content of an RFD	40
$\underline{\underline{\dim}}_{\mathrm{Mi}}(E, \Omega)$	the relative lower Minkowski dimension of an RFD	41
$\overline{\overline{\dim}}_{\mathrm{Mi}}(E, \Omega)$	the relative upper Minkowski dimension of an RFD	41
$\dim_{\mathrm{Mi}}(E, \Omega)$	the Minkowski dimension of an RFD	41
$\mathscr{P}_U(\zeta_{E,\Omega})$	the set of visible complex dimensions of an RFD	43
$\mathscr{P}(\zeta_{E,\Omega})$	the set of complex dimensions of an RFD	43
$\lvert \cdot \rvert_p$	the p-adic absolute value	48
$\mathbb{Q}_p$	the p-adic numbers	49

Preface

Fractal zeta functions were first introduced in the 1990s by Lapidus and various collaborators in order to study certain spectral problems in one-dimensional Euclidean space (e.g. [Lap91, Lap93, LM95]). These zeta functions provide a tool for relating the geometry of a set with spectrum of the Dirichlet operator which operates on that set. In the intervening quarter century, these results have been extended and generalized to describe the geometry more general spaces (e.g. [LL08, LvF13, LRŽ16, Wat17]). The principle novelties of this thesis are two-fold: first, the theory of global fractal zeta functions is generalized to homogeneous metric measure spaces; and second, a local theory of fractal zeta functions is introduced.

The structure of this thesis

Chapter 1 provides motivation for the work done in this thesis, beginning with Kac's classic question "Can one hear the shape of a drum?" [Kac66]. This chapter gives a rough outline of previous work on fractal zeta functions, and concludes with an example computation in the Euclidean setting.

Chapter 2 outlines essential notation and definitions. The material presented in this chapter is not novel, but its inclusion should help orient the reader and establish notational conventions used throughout.

Chapter 3 introduces the distance and tube zeta functions associated to subsets of metric spaces which satisfy certain homogeneity conditions. This chapter shows that these fractal zeta functions have many of the important properties which might be expected by analogy to the results in [LRŽ16]. The main novel results are generalized versions of the statements in [LRŽ16, Thm. 2.1.11], which

concern the domains of holomorphicity of fractal zeta functions. Finally, Section 3.4 introduces relative fractal drums, which are an useful computational tool.

Chapter 4 recalls the construction of the p-adic numbers, then gives several example computations in p-adic spaces. Spaces of p-adic numbers provide an interesting sandbox for applying the theory of fractal zeta functions over more general metric spaces—such spaces are quite regular (in the sense of Ahlfors, for example) and the structure of the metric on $\mathbb{Q}_p$ ensures that self-similar sets are all "lattice," a distinction which is important in previous works which study self-similar fractal strings on $\mathbb{R}$. Fractal zeta functions over sets of p-adic numbers may also be of greater interest via the connections between fractal zeta functions and the classic Riemann zeta function. It is worth noting that the results in this chapter are similar to those described in [LL08], though the techniques employed here are novel.

Chapter 5 introduces local fractal zeta functions. Local fractal zeta functions are a tool somewhat analogous to the zeta functions associated to a relative fractal drum. However, in contrast to the theory developed in previous work, the local fractal zeta functions do not rely on an *a priori* notion of ambient dimension, and therefore provide intrinsic information regarding the geometry of a metric space. The chapter finishes with several examples.

The thesis concludes in Chapter 6 with a discussion of several open problems.

Chapter 1

Introduction

1.1 Can one hear the shape of a drum?

In his classic paper [Kac66], Mark Kac poses the question "Can one hear the shape of a drum?"[1] For example, take the open unit disk

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

in $\mathbb{R}^2$ and imagine stretching a membrane across this disk so that it may vibrate freely in the interior, but is fixed on the boundary (see Figure 1.1). When this drum is struck, it will produce a sound. If a membrane is similarly stretched across another bounded open set Ω in $\mathbb{R}^2$, and this new drum produces a sound which is indistinguishable from that of the drum corresponding to the disk $\mathbb{D}$, must it necessarily be the case that Ω is isometric to the disk?

The analogous problem in one dimension provides an informative first step in understanding Kac's problem: suppose that a string of uniform density is stretched and attached to two pegs. If the string is plucked, it will vibrate and produce a tone. The vibrations of the string are modeled by solutions u to the wave equation, where $u(t, x)$ denotes the height of a point x on the string (above

[1] This question had likely been floating around the mathematical community for quite a long time before Kac wrote it down. Kac himself credits Bochner with introducing him to the problem [Kac66, p. 3].

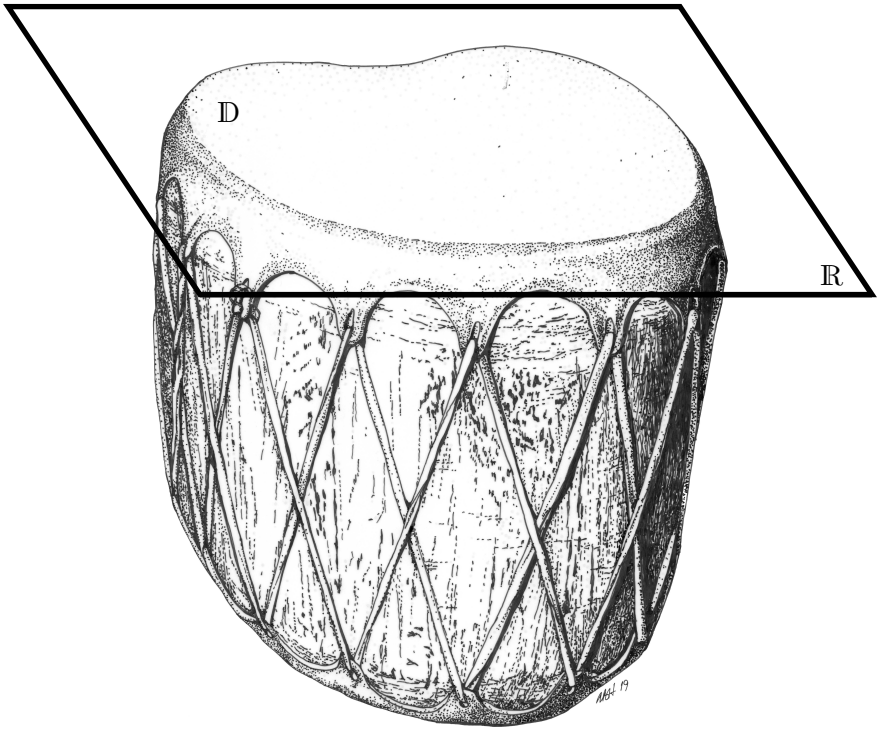

Figure 1.1: A compact set $\mathbb{D}$ in $\mathbb{R}^2$ can be imagined as the head of a drum.

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or below its rest position) at time t. That is, u solves the boundary value problem

$$\begin{cases} u_{tt}(t,x) - c^2 \Delta u(t,x) = 0 & \text{for } (t,x) \in [0,\infty) \times (0,L), \text{ and} \\ u(t,0) = u(t,L) = 0 & \text{for } t \in [0,\infty), \end{cases} \tag{1.1.1}$$

where the two pegs are at the points $x = 0$ and $x = L$, the symbol Δ denotes the Laplace operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, and c is a constant which depends on the density of the string and the tension acting along the string.

This equation can be solved via the technique of separation of variables. Under the *ansatz* that $u(t,x) = T(t)X(x)$ for some functions T and X, the differential equation becomes

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

where λ does not depend on either t or x, and is therefore a constant. This gives rise to the system of ordinary differential equations

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ T''(t) - \lambda c^2 T(t) = 0. \end{cases}$$

If $\lambda = 0$, then there are no nontrivial solutions which satisfy the boundary conditions given at (1.1.1). Otherwise, the first equation can be solved by making the additional *ansatz* that $X(x) = \mathrm{e}^{\sigma x}$ for some value σ. Under this assumption,

$$X''(x) - \lambda X(x) = 0 \implies \sigma^2 \mathrm{e}^{\sigma x} - \lambda \mathrm{e}^{\sigma x} = 0 \implies \sigma = \pm\sqrt{\lambda}.$$

There are no nontrivial solutions if $\lambda > 0$, but if $\lambda < 0$, then the assumed form of X gives rise to the family of solutions

$$X(t) = c_1 \mathrm{e}^{\mathbb{i}\sqrt{|\lambda|}x} + c_2 \mathrm{e}^{-\mathbb{i}\sqrt{|\lambda|}x}$$

where c_1 and c_2 are constants, and $\mathbb{i}^2 = -1$. It follows from the boundary conditions $X(0) = X(L) = 0$ that λ must satisfy

$$\sqrt{\lambda} = \frac{n\pi}{L} =: \omega_n$$

for some $n \in \mathbb{N}$.[2] Indeed, for each $n \in \mathbb{N}$, the function

$$X_n(x) = \frac{1}{2\mathbb{i}} \left(\mathrm{e}^{\mathbb{i}\omega_n x} - \mathrm{e}^{-\mathbb{i}\omega_n x} \right) = \sin(\omega_n x)$$

is an eigenfunction of the Laplace operator Δ with corresponding eigenvalue ω_n^2. Solutions to the first ordinary differential equation may be written as (possibly infinite) linear combinations of these eigenfunctions, and general solutions to the wave equation (1.1.1) may be written as (possibly

[2] Throughout this text, the symbol $\mathbb{N}$ denotes the set of natural numbers, which is the set of strictly positive integers.

infinite) linear combinations of functions of the form

$$u_n(t,x) = \big(a_n \cos(\omega_n ct) + b_n \sin(\omega_n ct)\big) \sin(\omega_n x),$$

where a_n and b_n are constants depending on an initial condition. The spectrum of eigenvalues corresponds to the "frequencies" of vibration—if the vibrating string is thought of as a string on a harp, this can be interpreted as the collection of tones and overtones which are heard when the string is plucked.

The eigenvalues of the Laplace operator uniquely determine the length of the interval, since

$$\omega_n = \frac{n\pi}{L} \implies L = \frac{n\pi}{\omega_n}.$$

In particular, the least eigenvalue ω_1 (the "fundamental frequency") determines the length of the string as

$$L = \frac{\pi}{\omega_1}.$$

In this setting, Kac's question becomes "If two strings are plucked and produce the same tone, must they be of the same length?", which has an affirmative answer. Perhaps more importantly, this example demonstrates that Kac's question may be rephrased as an inverse spectral problem: if the Laplace operators on two domains Ω_1 and Ω_2 are identical—that is, if Ω_1 and Ω_2 are isospectral—must they be congruent?

Continue to work in a slightly more general one-dimensional setting. Let Ω be a bounded open subset of $\mathbb{R}$, which may be written as the (possibly infinite) disjoint union of open intervals. If strings of the same uniform density are attached to the endpoints of each such interval and placed under the same uniform tension, then Ω becomes a harp. If the strings of this harp are all plucked, the vibration of the strings is modeled by solutions to the boundary value problem

$$\begin{cases} u_{tt}(t,x) - c^2\Delta(t,x) & \text{for } (t,x) \in [0,\infty) \times \Omega, \text{ and} \\ u(t,x) = 0 & \text{for } (t,x) \in [0,\infty) \times \partial\Omega. \end{cases}$$

As in the case of a single string, the "sound" produced by this harp can be described in terms of the eigenvalues (or, more generally, the spectrum) of the Laplace operator, i.e. the solutions to the Dirichlet eigenvalue problem

$$\begin{cases} \Delta X = \lambda X & \text{if } x \in \Omega, \text{ and} \\ X = 0 & \text{if } x \in \partial\Omega. \end{cases}$$

The corresponding inverse spectral problem is explored by [Lap93] and [LM95], wherein conditions under which the problem has an affirmative answer are described. A summary of the problem and relevant results can be found in [LvF13, Ch. 9].

1.2 Fractal Zeta Functions

The geometric zeta function associated to bounded open subset of $\mathbb{R}$ is an important tool in the study of the inverse spectral problem described above. In brief, a *fractal harp* $\mathcal{L}$ is a bounded, open subset of $\mathbb{R}$. Any such set can be written as the disjoint union of countably many open intervals of finite length, and any two such sets are isometric if they consist of intervals of the same lengths. Hence a fractal harp may be thought of as a sequence of lengths, i.e.

$$\mathcal{L} = (\ell_1, \ell_2, \ell_3, \dots),$$

where each $\ell_j \in \mathbb{R}$ is the length of one of the open intervals comprising $\mathcal{L}$. Define the *geometric zeta function* corresponding to a fractal harp $\mathcal{L}$ by

$$\zeta_{\mathcal{L}}(s) := \sum_{j=1}^{\infty} \ell_j^s,$$

where s is a complex variable. The geometric zeta function was introduced in order to study the Dirichlet operator on $\mathcal{L}$, but it is itself an object of interest.

In general, the series defining the geometric zeta function will converge on a right half-plane bounded by the "fractal dimension" of the *boundary* of $\mathcal{L}$, and will have a singularity at this dimension. Under relatively mild hypotheses on $\mathcal{L}$, the geometric zeta function will extend to

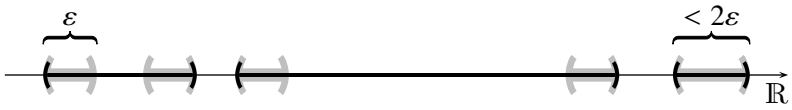

Figure 1.2: The harp $\mathcal{L}$ consists of the three open intervals shown in black, and a tubular neighborhood of radius ε is shown in grey. The first two intervals of $\mathcal{L}$ are longer than 2ε, and each contribute two intervals to the tubular neighborhood. The third interval has length less than 2ε, and so contributes only a single interval to the tubular neighborhood.

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a meromorphic function on a larger open domain which strictly contains the right half-plane of convergence. In this case, the singularity at the dimension will be a pole—in a meaningful sense, this pole *is* the fractal dimension of the fractal string. Other poles in the domain of meromorphic continuation also provide useful geometric information—for example, tube formulæ expressed in terms of the set of poles.

Let $\zeta_{\mathcal{L}}$ denote the geometric zeta function corresponding to a fractal string $\mathcal{L}$. Assume that $\zeta_{\mathcal{L}}$ extends to a meromorphic function on some "sufficiently large" domain, and let $\mathscr{P}(\zeta_{\mathcal{L}})$ denote the set of poles $\zeta_{\mathcal{L}}$ on that domain. For $\varepsilon > 0$, a *tubular neighborhood of the boundary of $\mathcal{L}$ with radius ε* is the collection of points in $\mathcal{L}$ which are less than ε units from the boundary—see Figure 1.2. Let $V(\varepsilon)$ denote the volume (one-dimensional Lebesgue measure) of such a tubular neighborhood. For example, if $\mathcal{L} = (0, 1)$, then

$$V(\varepsilon) = \begin{cases} 2\varepsilon & \text{if } \varepsilon < \frac{1}{2}, \text{ and} \\ 1 & \text{if } \varepsilon \geq \frac{1}{2}. \end{cases}$$

If $\mathcal{L}$ is sufficiently "well-behaved", then

$$V(\varepsilon) = \sum_{\omega \in \mathscr{P}(\zeta_{\mathcal{L}})} \operatorname{res}\left(\frac{\zeta_{\mathcal{L}}(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega \right) + \text{error}(\varepsilon),$$

where $\text{error}(\varepsilon)$ is an error term, which can be given explicitly in terms of ε. This and other volume formulæ are discussed in much greater detail in [LvF13, Ch. 8].

The geometric zeta function is a fundamentally "one-dimensional" tool—it does not directly generalize to higher dimensions. However, the observation that the volume of a tubular neighborhood of the boundary of a harp is given in terms of the geometric zeta function motivates generalizations to higher dimensional spaces.

Let E be a bounded subset of $\mathbb{R}^d$, and for any $\delta > 0$, let

$$E_\delta := \{x \in \mathbb{R}^d \mid d(x, E) < \delta\}$$

denote a δ-neighborhood of E. Define the *tube zeta function* $\tilde{\zeta}_E$ and the *distance zeta function* ζ_E corresponding to E by the integrals

$$\tilde{\zeta}_E(s) := \int_0^\delta t^{s-d-1}|E_t|\,\mathrm{d}t \qquad \text{and} \qquad \zeta_E(s) := \int_{E_\delta} d(x, E)^{s-d}\,\mathrm{d}x,$$

where $|E_t|$ denotes the Lebesgue measure of the set E_t. A change of variables relates the two zeta functions, with the precise functional relation being

$$\zeta_E(s) = \delta^{s-d}|E_\delta| + (d - s)\tilde{\zeta}_E(s).$$

From this presentation, it is apparent that the two zeta functions differ only be an entire function, hence they have similar analytic properties. These properties are analogous to those of the geometric zeta function. For example, the higher dimensional zeta functions converge on a half-plane to the right of the "fractal dimension" of E, and the volume of a tubular neighborhood of E can be expressed in terms of the residues of the tube zeta function.

Example 1.1. Roughly speaking, the Sierpinski carpet is the limiting object in a recursive construction. As a basis for the construction, let E_0 be the closed unit square in $\mathbb{R}^2$, i.e. the set

$$E_0 := [0, 1]^2.$$

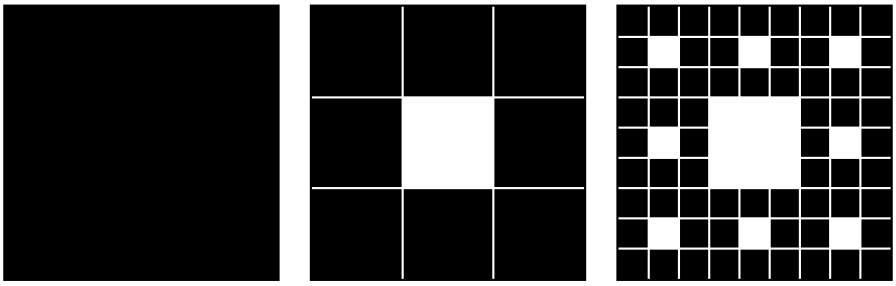


Figure 1.3: The first three stages in the construction of the Sierpinski carpet.

The set E_n is composed of 8^n squares with side length 3^{-n}. To obtain E_{n+1} from E_n, divide each of these 8^n squares into nine congruent squares of side length $3^{-(n+1)}$, and remove the center square. The first three sets obtained via this construction are shown in Figure 1.3. The Sierpinski carpet is the set

$$\mathcal{SC} = \bigcap_{j=0}^{\infty} E_j.$$

The distance zeta function associated to the Sierpinski carpet is given by

$$\zeta_{\mathcal{SC}}(s) = \int_{\mathcal{SC}_\delta} d(x, \mathcal{SC})^{s-2} \, \mathrm{d}x,$$

where it is convenient to assume that $\delta > \frac{1}{6}$. As shown in Figure 1.4, the δ-neighborhood $\mathcal{SC}_\delta$ may be decomposed into four quarter circles, four rectangles, and the union of the removed squares. Thus

$$\zeta_{\mathcal{SC}}(s) = \underbrace{4 \int_0^{\pi/2} \int_0^{\delta} (r^2)^{s-2} r \, \mathrm{d}r \, \mathrm{d}\theta}_{\text{quarter circles}} + \underbrace{4 \int_0^1 \int_0^{\delta} x_2^{s-2} \, \mathrm{d}x_2 \, \mathrm{d}x_1}_{\text{rectangles}} + \sum_{U=\text{removed square}} \int_U d(x, \mathcal{SC})^{s-2} \, \mathrm{d}x.$$

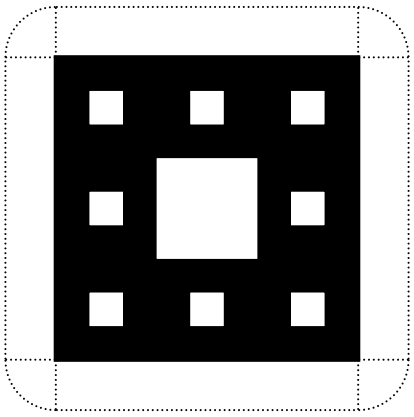


Figure 1.4: With $\delta > \frac{1}{6}$, a δ-neighborhood of the Sierpinski carpet may be decomposed into four quarter circles (at the outside corners), four rectangles (along the outside edges), and the union of squares removed in the construction.

Computing the first two integrals is an exercise in elementary multivariable calculus. Performing these computations simplifies the distance zeta function to

$$\zeta_{\mathcal{SC}}(s) = 2\pi\frac{\delta^s}{s} + 4\frac{\delta^{s-1}}{s-1} + \sum_{U=\text{removed square}} \int_U d(x,\mathcal{SC})^{s-2}\,\mathrm{d}x. \tag{1.2.1}$$

To compute the last integral, it is helpful to first determine what each removed square contributes to the distance zeta function.

Let U_n be one of the (open) squares removed in the n-th stage of the construction of the Sierpinski carpet, so that the length of each side of U_n is 3^{-n}. The goal now is to compute

$$\int_{U_n} d(x,\mathcal{SC})^{s-2}\,\mathrm{d}x.$$

Following the hint shown in Figure 1.5, this integral may be evaluated as

$$\int_{U_n} d(x,\mathcal{SC})^{s-2}\,\mathrm{d}x = 8\int_0^{3^{-n}/2}\int_0^{x_1} x_2^{s-2}\,\mathrm{d}x_2\,\mathrm{d}x_1 = \left(\frac{8}{s(s-1)}\right)\left(\frac{1}{2\cdot 3^n}\right)^s.$$

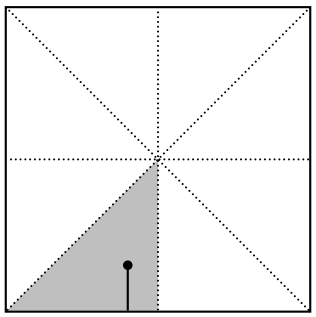


Figure 1.5: Orient a square of side length L so that the lower-left corner sits at the origin and so that the sides sit along the horizontal and vertical axes. The distance from a point in the shaded region to the boundary of the square is the height of the point above the horizontal axis, e.g. $d((x_1, x_2), \partial(\text{square})) = x_2$.

In the n-th stage of the construction, 8^n such squares are removed, implying that

$$\begin{aligned}\sum_{U=\text{removed square}} \int_U d(x, \mathcal{SC})^{s-2}\,\mathrm{d}x &= \sum_{n=1}^{\infty}\left(8^n \int_{U_n} d(x, \mathcal{SC})^{s-2}\,\mathrm{d}x\right)\\ &= \frac{8}{2^s s(s-1)}\sum_{n=1}^{\infty} 8^n\left(\frac{1}{3^n}\right)^s\\ &= \frac{2^{6-s}}{s(s-1)}\left(\frac{1}{3^s-8}\right),\end{aligned}$$

which gives the remaining term in the distance zeta function at (1.2.1).

The distance zeta function associated to the Sierpinski carpet is therefore given by

$$\zeta_{\mathcal{SC}}(s) = 2\pi\frac{\delta^s}{s} + 4\frac{\delta^{s-1}}{s-1} + \frac{2^{6-s}}{s(s-1)}\left(\frac{1}{3^s-8}\right).$$

This function is meromorphic on the entire complex plane ("mentire") and possesses simple poles at $s = 0$, 1 and for

$$s \in \frac{\log(8)}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)},$$

where $\mathbb{i}$ denotes the imaginary unit, which satisfies $\mathbb{i}^2 = -1$. ◁

Chapter 2

Background

2.1 Pushforward measures

The following definition and theorem are stated in [Bog06, §3.6].

Definition 2.1. Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be two measurable spaces, and let $f : X \to Y$ be an $(\mathscr{M}, \mathscr{N})$-measurable function. Then for any bounded (or bounded from below) measure μ on $\mathscr{M}$, the formula

$$f_*(\mu) : N \mapsto \mu\left(f^{-1}(N)\right), \qquad N \in \mathscr{N}$$

defines a measure on $\mathscr{N}$ called the *pushforward measure* of μ under f.

Theorem 2.2 ([Bog06, Thm. 3.6.1])**.** *Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be two measurable spaces, and let μ be a nonnegative measure on $\mathscr{M}$. An $\mathscr{N}$-measurable function g on Y is integrable with respect to the measure $f_*(\mu)$ if and only if the function $g \circ f$ is integrable with respect to μ. Moreover,*

$$\int_Y g(y)\, \mathrm{d}(f_*(\mu))(y) = \int_X (g \circ f)(x)\, \mathrm{d}\mu(x). \tag{2.1.1}$$

If X and Y are Euclidean spaces (e.g. $X = Y = \mathbb{R}^d$), then Theorem 2.2 reduces to the usual change of variables formula, thus (2.1.1) provides a more general change of variables formula for measure spaces.

2.2 Notions of dimension

Throughout this section, (X, d, μ) is metric measure space with complete metric d, and Borel regular measure μ. Additionally, the measure has the property that each ball has finite, nonzero measure, that is,

$$0 < \mu(B(x, r)) < \infty$$

for all $x \in X$ and finite $r > 0$.

Definition 2.3. For a fixed $q \geq 0$, the space X is *Ahlfors q-regular* (or simply *Ahlfors regular*) if there exists a constant $M > 0$ such that

$$\frac{1}{M} r^q \leq \mu(B(x, r)) \leq M r^q$$

for all $x \in X$ and $r > 0$.

The Ahlfors regularity condition essentially states that all balls in X scale in the same manner (up to to a multiplicative constant). Ahlfors regular spaces are then extremely homogeneous in the sense that the scaling properties of a ball are independent both of the ball's location in the space and its radius.

Definition 2.4. For a fixed $q \geq 0$, the measure μ is *q-homogeneous* on X if there is a constant $M > 0$ such that

$$\frac{\mu(B(x, r))}{r^q} \leq M \frac{\mu(B(\xi, \rho))}{\rho^q}$$

for all $0 < \rho < r$, $x \in X$, and $\xi \in B(x, r)$. If there is some $q < \infty$ such that μ is q-homogenous on X, then μ is *homogeneous* on X.

Definition 2.5. The *measure theoretic Assouad dimension* of (X, d, μ) is defined to be

$$\dim_{\mathrm{As}}(X, d, \mu) := \inf \{q \mid \mu \text{ is } q\text{-homogenous on } X\} .$$

If there is no danger of ambiguity, omit the metric and measure from the notation, and write $\dim_{\mathrm{As}}(X)$.

It appears that this notion of dimension was introduced by Lehrbäck and Tuominen in [LT13], where it is called the "doubling dimension" of X. Their nomenclature stems from the observation that if a measure has the doubling property, i.e. if there is a "doubling constant" M_d such that

$$\mu(B(x,2r)) \leq M_d \mu(B(x,r))$$

for all $x \in X$ and $r > 0$, then μ will be q-homogeneous with $q = \log_2(M_d)$. The doubling constant gives an upper bound on the dimension of a space—indeed, a measure is doubling if and only if it is homogeneous.

Unfortunately, authors use the term "doubling dimension" inconsistently in the literature on metric spaces. Instead of using this potentially ambiguous term, adopt the phrase "measure theoretic Assouad dimension" for the notion presented in Definition 2.5. The construction here is in terms of the measures of balls, which parallels the usual defintion of the Assouad dimension in terms of ball counting functions.[1]

Definition 2.6. Let $E \subseteq X$ be bounded. Then for $t > 0$, the *t-neighborhood* of E is the set

$$E_t := \{x \in X \mid d(x,E) < t\}$$

Definition 2.7. Suppose that $\dim_{\mathrm{As}}(X) = Q < \infty$ and let $E \subseteq X$ be bounded. For $q \geq 0$, the *q-dimensional lower Minkowski content* of E is

$$\underline{\mathfrak{M}}^q(E) := \liminf_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-q}}.$$

Similarly, the *s-dimensional upper Minkowski content* of E is

$$\overline{\mathfrak{M}}^q(E) := \limsup_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-q}}.$$

[1] As a historical note, the notion that is now commonly referred to as the Assouad dimension was originally introduced by Bouligand [Bou28], but received little attention at the time. It was reintroduced much later by Assouad [Ass79] in his 1979 thesis, where it was used to study certain embedding problems.

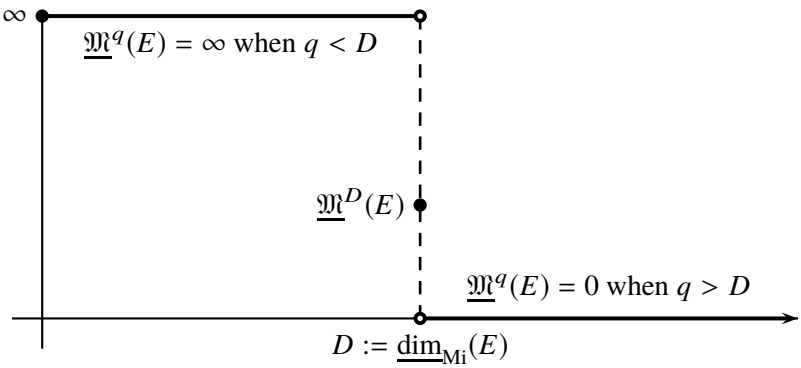


Figure 2.1: The graph of $\underline{\mathfrak{M}}^q(E)$ as a function of q. With $D := \underline{\dim}_{\mathrm{Mi}}(E)$, the q-dimensional lower Minkowski content of E is infinite whenever q is less than D, and zero whenever q is greater than D. When $q = D$, the lower Minkowski content of E can take any value (depending on E). A similar result holds for $\overline{\mathfrak{M}}^q(E)$.

Observe that there is generally a "natural" dimension in which to measure the (upper or lower) Minkowski content of a set. Suppose that $\underline{\mathfrak{M}}^D(E) < \infty$ and that $q > D$. Then

$$\underline{\mathfrak{M}}^q(E) = \liminf_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-q}} = \liminf_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-D}} t^{q-D} = \underline{\mathfrak{M}}^q(E) \liminf_{t \searrow 0} t^{q-D} = 0.$$

Similarly, if $\underline{\mathfrak{M}}^D(E)$ is finite, then $\underline{\mathfrak{M}}^q(E)$ will be infinite for any $q < D$. Hence there is a unique (possibly infinite) value D which such that the q-dimensional lower Minkowski content is zero for q greater than D, and infinite for q less than D, see Figure 2.1. A similar result holds for the upper Minkowski content, giving rise to the notion of Minkowski dimension:

Definition 2.8. Suppose that $\dim_{\mathrm{As}}(X) = Q < \infty$ and let $E \subseteq X$ be bounded. The *lower and upper Minkowski dimensions* of A are defined to be

$$\underline{\dim}_{\mathrm{Mi}}(E) := \inf\left\{s \,\middle|\, \underline{\mathfrak{M}}^q(E) = 0\right\}, \quad \text{and} \quad \overline{\dim}_{\mathrm{Mi}}(E) := \inf\left\{s \,\middle|\, \overline{\mathfrak{M}}^q(E) = 0\right\},$$

respectively. If $\underline{\dim}_{\mathrm{Mi}}(E) = \overline{\dim}_{\mathrm{Mi}}(E)$, then the *Minkowski dimension* of E is defined to be the common value, denoted $\dim_{\mathrm{Mi}}(E)$.

Equivalently, the lower and upper Minkowski dimensions may be defined as

$$\underline{\dim}_{\mathrm{Mi}}(E) := \sup\left\{s \,\middle|\, \underline{\mathfrak{M}}^q(E) = +\infty\right\} \quad \text{and} \quad \overline{\dim}_{\mathrm{Mi}}(E) := \sup\left\{s \,\middle|\, \overline{\mathfrak{M}}^q(E) = +\infty\right\},$$

Both characterizations of the Minkowski dimensions are used in the sequel. In particular, if $\underline{\mathfrak{M}}^q(E)$ (or $\overline{\mathfrak{M}}^q(E)$, resp.) is finite and nonzero, then $\underline{\dim}_{\mathrm{Mi}}(E) = q$ (or $\overline{\dim}_{\mathrm{Mi}}(E) = q$, resp.).

Note that the converse does not hold—there are examples of sets with lower (upper, resp.) Minkowski dimension q and either zero or infinite q-dimensional lower (upper, resp.) Minkowski content. For instance, a countably infinite, uniformly discrete subset of $\mathbb{R}^d$ (such as $\mathbb{Z}^d$) will have lower Minkowski dimension zero, but has infinite 0-dimensional lower Minkowski content.

Fix some $\varepsilon > 0$ and note that if $E \subseteq X$ is bounded, then so too is E_ε. By assumption, all balls have finite measure, and so $\mu(E_\varepsilon) < \infty$. With $Q = \dim_{\mathrm{As}}(X)$, take limits over $t < \varepsilon$ to obtain

$$\overline{\mathfrak{M}}^q(E) = \limsup_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-q}} \leq \limsup_{t \searrow 0} \frac{\mu(E_\varepsilon)}{t^{Q-q}} = \mu(E_\varepsilon) \cdot \limsup_{t \searrow 0} t^{q-Q} = 0$$

whenever $q > Q$. The measure theoretic Assouad dimension of the space X gives an upper bound for the upper Minkowski dimension of bounded subsets of X. As the upper Minkowski dimension bounds the lower Minkowski dimension, the relations

$$\underline{\dim}_{\mathrm{Mi}}(E) \leq \overline{\dim}_{\mathrm{Mi}}(E) \leq \dim_{\mathrm{As}}(X)$$

hold for any bounded $E \subseteq X$. While each of the inequalities may be strict, the following lemma gives an important case in which equality holds throughout.

Lemma 2.9. *Suppose that* $\dim_{\mathrm{As}}(X) = Q < \infty$ *and let* $E \subseteq X$ *be bounded. If* $\mu(\overline{E}) > 0$ *then*

$$\underline{\dim}_{\mathrm{Mi}}(E) = \overline{\dim}_{\mathrm{Mi}}(E) = \dim_{\mathrm{As}}(X).$$

In particular, the Minkowski dimension exists and is equal to the ambient measure theoretic Assouad dimension of the space.

Proof. The proof is by contraposition. Suppose that

$$\underline{\dim}_{\mathrm{Mi}}(E) < Q,$$

which implies that $\underline{\dim}_{\mathrm{Mi}}(E) = 0$. The map $t \mapsto \mu(E_t)$ is a nondecreasing function of t and is bounded from below by zero. This implies that $\lim_{t \searrow 0} \mu(E_t)$ exists. But then

$$\lim_{t \searrow 0} \mu(E_t) = \liminf_{t \searrow 0} \mu(E_t) = \liminf_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-Q}} = \underline{\mathfrak{M}}^{Q}(E) = 0.$$

As $\overline{E} \subseteq E_t$ for all $t > 0$, the monotonicity of the measure implies that $\mu(\overline{E}) \leq \mu(E_t)$ for all such t. Take limits to obtain

$$\mu(\overline{E}) = \lim_{t \searrow 0} \mu(\overline{E}) \leq \lim_{t \searrow 0} \mu(E_t) = 0.$$

Therefore if the lower Minkowski dimension of E is strictly less than measure theoretic Assouad dimension of the ambient space, then $\mu(\overline{E}) = 0$, which is the desired result. □

2.3 Iterated function systems and self-similar measures

Iterated function systems

Definition 2.10. An *iterated function system* (or *IFS*) on a metric space (X, d) is a collection of functions $\{\varphi_i : X \to X\}_{i \in \mathscr{I}}$ indexed by a finite set $\mathscr{I}$. Associated to each IFS, there is a map of sets $\Phi : 2^X \to 2^X$, defined by

$$\Phi(E) = \bigcup_{i \in \mathscr{I}} \varphi_i(E),$$

where 2^X denotes the powerset of X. By a slight abuse of notation, write $\Phi = \{\varphi_i\}_{i \in \mathscr{I}}$.

Definition 2.11. Let (X, d) be a metric space. A function $\varphi : X \to X$ is a *contraction mapping* if there is some constant $c \in (0, 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq c d(x, y) \tag{2.3.1}$$

for all $x, y \in X$. If equality holds in (2.3.1), then φ is a *contractive similitude* with *contraction ratio* c.

Definition 2.12. Let $\Phi = \{\varphi_i\}_{i \in \mathscr{I}}$ be an iterated function system. Then Φ is a *contractive iterated function system* (or *CIFS*) if φ_i is a contraction mapping for each $i \in \mathscr{I}$, and a *self-similar iterated function system* (or *SSIFS*) if φ_i is a contracting similitude for each $i \in \mathscr{I}$.

Theorem 2.13 ([Hut81, Thm. 3.1.3])**.** *Let (X, d) be a complete metric space and $\Phi = \{\varphi_i\}_{i \in \mathscr{I}}$ a CIFS on X. Then there is a unique nonempty compact set $\mathscr{A}$ such that*

$$\mathscr{A} = \bigcup_{i \in \mathscr{I}} \Phi(\mathscr{A}). \tag{2.3.2}$$

This set $\mathscr{A}$ is the attractor *(or* invariant set*) of* Φ.

When working with iterated function systems, it is useful to have a language to describe points or sets that occur as the image of compositions of maps from the system. To this end, adopt the notation and terminology outlined below.

Definition 2.14. Let $\mathscr{I}$ be a finite index set, and denote by $\mathscr{I}^*$ the collection of all finite tuples (or words) with entries in $\mathscr{I}$. More precisely,

$$\mathscr{I}^* := \bigcup_{k=0}^{\infty} \mathscr{I}^k,$$

where $\mathscr{I}^0 := \{\iota\}$ is the set containing the empty word, and $\mathscr{I}^k$ is the k-fold Cartesian product of $\mathscr{I}$ with itself.

Definition 2.15. Let $\{\varphi_i\}_{i \in \mathscr{I}}$ be an iterated function system, let $\boldsymbol{i} \in \mathscr{I}^*$, and let $|\boldsymbol{i}|$ denote the length of $\boldsymbol{i}$. Define

$$\varphi_{\boldsymbol{i}} := \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_{|\boldsymbol{i}|}}.$$

If the map φ_i is to be composed with itself n times, then simplify the notation by writing

$$\varphi_i^n := \varphi_{\boldsymbol{i}},$$

where $|\boldsymbol{i}| = n$ and $i_k = i$ for all $k = 1, 2, \ldots, n$.

Remark 2.16. In the study of iterated function systems, the indexing can sometimes become quite complicated. To limit the complexity, the symbol $\mathbb{i}$ is used for the imaginary unit—that is, $\mathbb{i} \in \mathbb{C}$ is defined to be the principal square root of -1. The symbols i and $\boldsymbol{i}$ are reserved for indexing.

Iterated function systems and their attractors have been well-studied in the literature on fractal geometry and dimension theory. In his now classic 1981 paper [Hut81], Hutchinson showed that if $\{\varphi_i\}_{i \in \mathscr{I}}$ is an SSIFS and there exists an open set U such that

(a) $\varphi_i(U) \subseteq U$ for all $i \in \mathscr{I}$, and

(b) $\varphi_i(U) \cap \varphi_j(U) = \varnothing$ for all $i \neq j$,

then the Hausdorff dimension of the attractor of the SSIFS is the unique solution D to the Moran equation

$$\sum_{i \in \mathscr{I}} c_i^D, \tag{2.3.3}$$

where c_i is the contraction ratio of φ_i. The conditions (a) and (b) constitute the *open set condition*, which is a kind of separation condition for an IFS.

In the time since Hutchinson codified the concept of an iterated function system, other authors have generalized his results by weakening the structure of the IFSes considered (e.g., McMullen [McM84] considers self-affine, rather than self-similar sets), by weakening the separation conditions (e.g., Lau and Ngai [LN99] obtain results by placing "weak separation" conditions on the monoid structure of an IFS, and Zerner [Zer96] studies various separation conditions related to the weak separation of Lau and Ngai), or by considering other notions of dimension (e.g. Mackay [Mac11] considers the Assouad dimension of certain self-affine sets).

Of particular interest in the current setting is a result of Fraser *et al.* [FHOR15], who show that if $\mathscr{A}$ is the attractor of a SSIFS which satisfies the open set condition, then

$$\dim_{\mathrm{H}}(\mathscr{A}) = \dim_{\mathrm{Mi}}(\mathscr{A}) = \dim_{\mathrm{As}}(\mathscr{A}) = D,$$

where D is again the unique real solution to the Moran equation (2.3.3). These sets are highly regular, and provide a useful collection of examples with know properties.

Self-similar measures

Definition 2.17. Let $\Phi = \{\varphi_i\}_{i\in\mathscr{I}}$ be an SSIFS on a complete metric space (X, d) with attractor $\mathscr{A}$. Further suppose that

$$\varphi_i(\mathscr{A}) \cap \varphi_j(\mathscr{A}) = \varnothing$$

whenever $i \neq j$. To each map φ_i, associate a probability or weight $\mathfrak{p}_i \in (0, 1)$ such that

$$\sum_{i\in\mathscr{I}} \mathfrak{p}_i = 1.$$

The collection of pairs $\{(\varphi_i, \mathfrak{p}_i)\}_{i\in\mathscr{I}}$ is a *weighted SSIFS*. If the context is unambiguous, write

$$(\Phi, \mathfrak{p}) := \{(\varphi_i, \mathfrak{p}_i)\}_{i\in\mathscr{I}},$$

where $\mathfrak{p}$ denotes the set of weights $\{\mathfrak{p}_i\}_{i\in\mathscr{I}}$.

Theorem 2.18. *Let $(\Phi, \mathfrak{p})$ be a weighted SSIFS indexed by $\mathscr{I}$. Let $\mathscr{A}$ denote the attractor of Φ and suppose that Φ is strongly separated in the sense that*

$$\varphi_i(\mathscr{A}) \cap \varphi_j(\mathscr{A}) = \varnothing$$

whenever $i \neq j$. There is a measure $\mu_{\mathfrak{p}}$ supported on $\mathscr{A}$ such that

$$\mu_{\mathfrak{p}}(\varphi_{\boldsymbol{i}}(\mathscr{A})) = \mathfrak{p}_{\boldsymbol{i}} := \prod_{k=1}^{|\boldsymbol{i}|} \mathfrak{p}_k. \tag{2.3.4}$$

This construction parallels that given by [Fal04, §17.3], with proof of Theorem 2.18 following by the argument outlined in [Fal04, Prop. 1.7]. It is worth noting that this measure is self-similar in

the sense that

$$\mu_{\mathfrak{p}}(E) = \sum_{i \in \mathscr{I}} \mathfrak{p}_i \mu_{\mathfrak{p}}(\varphi_i^{-1}(E))$$

for any $\mu_{\mathfrak{p}}$-measurable set E. Hence if $(\Phi, \mathfrak{p})$ is a weighted SSIFS satisfying the hypotheses of Theorem 2.18, refer to the corresponding measure as a *self-similar measure*.

Self-similar measures are an example of a broader class of measures called *multifractal measures* (or simply *multifractals*). As is the case with the term "fractal," the term "multifractal" has no widely agreed upon definition. However, multifractal measures are generally characterized by their variable homogeneity—the nature of the scaling relationship between distance and volume can vary radically from point to point in the space. The inhomogeneity of a multifractal measure can be described in terms of the scaling behaviour of families of balls with a common center, which corresponds to a "local" notion of dimension.

Definition 2.19. Let (X, d, μ) be a complete metric measure space. The *lower and upper local dimensions* of the measure μ at $x \in X$ are

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r \searrow 0} \frac{\log(\mu(B(x,r)))}{\log(r)}$$

and

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r \searrow 0} \frac{\log(\mu(B(x,r)))}{\log(r)},$$

respectively. When $\underline{\dim}_{\mathrm{loc}}\mu(x) = \overline{\dim}_{\mathrm{loc}}\mu(x)$, the *local dimension* of μ at x, denoted $\dim_{\mathrm{loc}}\mu(x)$, is the common value.

2.4 Mentire functions

The main objects of study in this thesis are fractal zeta functions, which are complex-valued functions of a single complex variable. These functions often possess "nice" analytic properties, but typically fail to be entire (that is, they typically fail to be holomorphic on the entire complex plane). Indeed, it is the poles of these functions which provide information about their associated spaces.

Many of the examples discussed in the sequel will involve fractal zeta functions which are meromorphic on $\mathbb{C}$. For brevity, these functions are "mentire" (a contraction of the description "**m**eromorphic on the **entire** complex plane"). More formally,

Definition 2.20. A function $f : \mathbb{C} \to \mathbb{C}$ is *mentire* if it is meromorphic on $\mathbb{C}$.

Chapter 3

Fractal Zeta Functions

Lapidus *et al.* develop a theory of zeta functions associated to bounded sets and "relative fractal drums" in Euclidean space [LRŽ16]. These zeta functions encode much of the geometry of the corresponding sets and drums, including notions of dimension and volume. This chapter extends the basic theory of fractal zeta functions to metric spaces which carry homogeneous measures.

Throughout this chapter, let (X, d, μ) be a metric measure space with complete metric d and positive Radon measure μ.[1] Assume further that μ is homogenous on X, with

$$Q = \dim_{\mathrm{As}}(X).$$

3.1 The distance zeta function

This section begins with a definition of the distance zeta function associated to a bounded subset of X. Initially, the distance zeta function is defined formally by an integral. The main goal of the remainder of the section is to show that this formal definition "makes sense" and gives a function on an appropriate domain in $\mathbb{C}$.

[1] Most of the results in this chapter are easily extended from positive Radon measures to signed and complex Radon measures. The arguments are essentially the same, but involve some technicalities which have the potential to obscure the main ideas. As such, the discussion is restricted to the positive case.

Definition 3.1. Let $E \subseteq X$ be bounded. The *distance zeta function* associated to E is the complex valued function $\zeta_{E,E_\delta} : U \to \mathbb{C}$ defined by the integral

$$\zeta_{E,E_\delta}(s) := \int_{E_\delta} d(x,E)^{s-Q} \, \mathrm{d}\mu(x), \tag{3.1.1}$$

where $\delta > 0$, and $U \subseteq \mathbb{C}$ is an appropriate domain.

At this point in the exposition, the distance zeta function associated to a bounded set $E \subseteq X$ is entirely formal, as the meaning of "appropriate domain" is not immediately obvious. Indeed, it is not even apparent that there is any domain on which the integral (3.1.1) defines a function. Even if the integral gives rise to a well-defined function for some fixed $\delta > 0$, a different choice of δ may define a radically different function. The remainder of this section addresses these issues.

Suppose that the integral in (3.1.1) is defined on some domain U. As noted above, "the" distance zeta function associated to E depends on a choice of δ. Observe that if $\delta' > \delta > 0$, then

$$\begin{aligned}
\zeta_{E,E_{\delta'}}(s) &= \int_{E_{\delta'}} d(x,E)^{s-Q} \, \mathrm{d}\mu(x) \\
&= \underbrace{\int_{E_{\delta'} \setminus E_\delta} d(x,E)^{s-Q} \, \mathrm{d}\mu(x)}_{=:\xi(s)} + \int_{E_\delta} d(x,E)^{s-Q} \, \mathrm{d}\mu(x) \\
&= \xi(s) + \zeta_{E,E_\delta}(s),
\end{aligned}$$

where s ranges over U. If $x \in E_{\delta'} \setminus E_\delta$, then $d(x,E) \in [\delta, \delta')$. Thus for any fixed $s \in \mathbb{C}$, the integrand is bounded by

$$\left| d(x,E)^{s-Q} \right| = d(x,E)^{\Re(s)-Q} \leq (\max\{\delta', \delta\})^{\Re(s)-Q} =: C\mathrm{e}^{\Re(s)}, \tag{3.1.2}$$

where the value of the maximum (and therefore the value of the constant C) depends only on the sign of the exponent. The inequality (3.1.2) implies that

$$\int_{E_{\delta'} \setminus E_\delta} \left| d(x,E)^{s-Q} \right| \, \mathrm{d}\mu(x) \leq \int_{E_{\delta'} \setminus E_\delta} C\mathrm{e}^{\Re(s)} \, \mathrm{d}\mu(x) = C\mathrm{e}^{\Re(s)} \mu(E_{\delta'} \setminus E_\delta),$$

which is finite by the hypothesis that E is bounded and the assumption that balls of finite radius have finite measure. Standard arguments imply that ξ extends to a function which is holomorphic on $\mathbb{C}$ (see, for example, [Sim15, Thm. 3.1.6]), and so $\zeta_{E,E_{\delta'}}$ and ζ_{E,E_δ} differ only by an entire function. As the ultimate goal is to study the analytic properties of the distance zeta function (e.g. the domain of holomorphicity, the poles of meromorphic extensions, and so on), the dependence on a choice of δ is inessential. These remarks are summarized in the following lemma:

Lemma 3.2. *Let E be a bounded subset of X. If $\delta' > \delta > 0$, then*

$$\zeta_{E,E_{\delta'}}(s) = \xi(s) + \zeta_{E,E_\delta}(s),$$

where ξ is an entire function.

Proof. See the preceding discussion. □

Lemma 3.2 permits a simplification of notation. If $E \subseteq X$ is bounded and $\delta > 0$, then the distance zeta function corresponding to E is

$$\zeta_E(s) := \zeta_{E,E_\delta}(s) = \int_{E_\delta} d(x,E)^{s-Q}\,\mathrm{d}\mu(x).$$

By a slight abuse of terminology, the distance zeta function $\zeta_E(s)$ is said to converge if and only if the defining integral converges absolutely (in the sense of Lebesgue).

The next several results describe the regions on which the distance zeta function converges and diverges, in the sense just introduced. These results, summarized in Remark 3.7 below, make it possible to give meaning to the phrase "appropriate domain" in Definition 3.1.

The first step is to demonstrate that the integral defining the distance zeta function converges on an open half-plane bounded in terms of the upper Minkowski dimension of E. This is done in Lemma 3.3, which establishes an estimate attributed to Harvey and Polking [HP70]. The version of the estimate given here is a generalization of [LRŽ16, Lemma 2.1.3], and is proved in a similar manner.

Lemma 3.3 (Harvey-Polking Estimate)**.** *Let $E \subseteq X$ be bounded and fix $\delta > 0$. If $\sigma < Q - \overline{\dim}_{\mathrm{Mi}}(E)$, then*

$$\int_{E_\delta} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) < \infty. \tag{3.1.3}$$

In particular, this implies that $\zeta_E(s)$ converges for all $\sigma > \overline{\dim}_{\mathrm{Mi}}(E)$.

Proof. Under certain additional hypotheses, the desired result follows almost immediately from the appropriate definitions:

- Suppose that $\sigma \leq 0$. The function $x \mapsto d(x,E)^{-\sigma}$ is continuous on E_δ. If $x \in E_\delta$, then $d(x,E) < \delta$, which implies that $d(x,E)^{-\sigma} \leq \delta^{-\sigma}$. Integrate to obtain the bound

$$\int_{E_\delta} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) \leq \int_{E_\delta} \delta^{-\sigma} \, \mathrm{d}\mu(x) = \delta^{-\sigma} \mu(E_\delta) < \infty,$$

  and so (3.1.3) holds whenever $\sigma \leq 0$.

- Suppose that $\mu(\overline{E}) > 0$. Lemma 2.9 implies that $Q - \overline{\dim}_{\mathrm{Mi}}(E) = 0$, hence $\sigma < Q - \overline{\dim}_{\mathrm{Mi}}(E) = 0$. The estimate (3.1.3) holds by the previous argument.

Suppose now that $\sigma < 0$ and that $\mu(\overline{E}) = 0$. Since $\mu(\overline{E}) = 0$ and the integral of any function over a null set is zero,

$$\int_{E_\delta} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) = \int_{E_\delta \setminus \overline{E}} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x).$$

Decompose $E_\delta \setminus \overline{E}$ into dyadic annular neighborhoods: for each $j \in \mathbb{N}$, let

$$A_j := E_{2^{-j}\delta} \setminus E_{2^{-(j+1)}\delta} = \left\{ x \in X \;\middle|\; 2^{-(j+1)}\delta \leq d(x,E) < 2^{-j}\delta \right\}$$

denote an annular neighborhood of E with outer radius $2^{-j}\delta$ and inner radius $2^{-(j+1)}\delta$. Note that there is a slight collision of notation here: the subscript j indicates an indexing of the annular

neighborhoods and not a j-neighborhood of some set A. The set E_δ is the disjoint union

$$E_\delta = \bigsqcup_{j=0}^{\infty} A_j.$$

Countable additivity of the measure implies that

$$\int_{E_\delta \setminus \overline{E}} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) = \sum_{j=0}^{\infty} \int_{A_j} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x)$$

If x is in A_j, then

$$2^{\sigma j}\delta^{-\sigma} < d(x,E)^{-\sigma} \leq 2^{\sigma(j+1)}\delta^{-\sigma},$$

since $\sigma > 0$. The integrands are therefore bounded on each annular neighborhood, from which follows the estimate

$$\begin{aligned}
\sum_{j=0}^{\infty} \int_{A_j} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) &\leq \sum_{j=0}^{\infty} \int_{A_j} 2^{\sigma(j+1)}\delta^{-\sigma} \, \mathrm{d}\mu(x) \\
&= \sum_{j=0}^{\infty} 2^{\sigma(j+1)}\delta^{-\sigma}\mu(A_j) \\
&= 2^{\sigma}\delta^{-\sigma} \sum_{j=0}^{\infty} 2^{\sigma j}\mu(A_j).
\end{aligned}$$

By assumption, $\sigma < Q - \overline{\dim}_{\mathrm{Mi}}(E)$, thus the interval $(\overline{\dim}_{\mathrm{Mi}}(E), Q-\sigma)$ is nonempty. Fix some q in this interval and observe that since q exceeds the upper Minkowski dimension of E, the q-dimensional upper Minkowski content of E must be zero. More precisely,

$$\overline{\mathfrak{M}}^q(E) = \limsup_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-q}} = 0,$$

and so there is a constant C depending only on δ such that $\mu(E_t) \leq Ct^{Q-q}$ for all $t \in (0,\delta]$. Take $t = 2^{-j}\delta$ to obtain the estimate

$$\mu(A_j) \leq \mu(E_{2^{-j}\delta}) \leq C\left(2^{-j}\delta\right)^{Q-q}$$

for all $j \in \mathbb{N}$, hence

$$2^{\sigma}\delta^{-\sigma}\sum_{j=0}^{\infty}2^{\sigma j}\mu(A_j) \leq C2^{\sigma}\delta^{Q-q-\sigma}\sum_{j=0}^{\infty}\left(2^{\sigma+q-Q}\right)^{j}.$$

As $q < Q - \sigma$, it follows that $2^{\sigma+q-Q} < 1$, and so the geometric series in the last line is absolutely convergent. Therefore

$$\int_{E_\delta} d(x,E)^{-\sigma}\,\mathrm{d}\mu(x) \leq C2^{\sigma}\delta^{Q-q-\sigma}\sum_{j=0}^{\infty}\left(2^{\sigma+q-Q}\right)^{j} < \infty.$$

In particular, (3.1.3) holds for all $\sigma < Q - \overline{\dim}_{\mathrm{Mi}}(E)$, which is the desired result. □

The Harvey-Polking estimate establishes the important fact that $\zeta_E(s)$ converges whenever s is a real number which exceeds the upper Minkowski dimension of E, that is,

$$\zeta_E(s) = \int_{E_\delta} d(x,E)^{s-Q}\,\mathrm{d}\mu < \infty$$

for all real $s > \overline{\dim}_{\mathrm{Mi}}(E)$. The goal now is to show that this lower bound on s is tight, in a sense made rigorous in Lemma 3.5. Lemma 3.5 is a version of [LRŽ16, Lemma 2.1.6], restated in the more general context of metric spaces which carry Radon measures, and proved using identical techniques. The proof of this lemma relies a techical result, which is recalled in Lemma 3.4.

Lemma 3.4 ([Fol99, Prop. 6.24])**.** *If $\sigma \in (0,\infty)$ and $f : X \to [0,\infty]$ then*

$$\int_X f^{\sigma}\,\mathrm{d}\mu = \sigma\int_0^{\infty} t^{-\sigma-1}\,\mu\left(\{x \mid f(x) > t\}\right)\,\mathrm{d}t.$$

Lemma 3.5. *Let $E \subseteq X$ be bounded and fix $\delta > 0$. If $\sigma > Q - \overline{\dim}_{\mathrm{Mi}}(E)$ then*

$$\int_{E_\delta} d(x,E)^{-\sigma}\,\mathrm{d}\mu(x) = +\infty.$$

Proof. Define $I : (0,\infty) \to [0,\infty]$ by

$$I(r) := \int_{E_r} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x).$$

The distance function is nonnegative, so if σ is fixed and $r' > r > 0$, then

$$I(r') = \left[\int_{E_{r'} \setminus E_r} + \int_{E_r} \right] d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) \geq \int_{E_r} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) = I(r).$$

The function I is nondecreasing in r. For any $r > 0$, take $f = d(\cdot,E)^{-1} \chi_{E_r}$ in Lemma 3.4 to obtain

$$\begin{aligned}
I(r) &= \int_{E_r} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) \\
&= \sigma \int_0^\infty t^{\sigma-1} \mu\left(\{x \mid d(x,E)^{-1} > t\} \cap E_r \right) \mathrm{d}t \\
&= \sigma \int_0^\infty t^{\sigma-1} \mu\left(E_{1/t} \cap E_\delta \right) \mathrm{d}t \\
&= \sigma \int_0^{1/r} t^{\sigma-1} \mu(E_r) \, \mathrm{d}t + \sigma \int_{1/r}^\infty t^{\sigma-1} \mu(E_{1/t}) \, \mathrm{d}t \\
&\geq \sigma \mu(E_r) \int_0^{1/r} t^{\sigma-1} \, \mathrm{d}t \\
&= \sigma \mu(E_r) \frac{1}{\sigma} t^\sigma \Big|_{t=0}^{1/r} \\
&= r^{-\sigma} \mu(E_r).
\end{aligned} \tag{3.1.4}$$

By hypothesis, $\overline{\dim}_{\mathrm{Mi}}(E) > Q - \sigma$, and so the interval $(Q-\sigma, \overline{\dim}_{\mathrm{Mi}}(E))$ is nonempty. For any q in this interval

$$\infty = \overline{\mathfrak{M}}^q(E) = \limsup_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-q}}.$$

By defintion of the lim sup, there is a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \searrow 0$ and

$$C_k := \frac{\mu(E_{t_k})}{t_k^{Q-q}} \to \infty \qquad \text{as } k \to \infty.$$

With $r = t_k$, the identity (3.1.4) is

$$I(t_k) \geq t_k^{-\sigma} \mu(E_{t_k}) = C_k t_k^{Q-q-\sigma}.$$

The C_k have been constructed so that $\lim_{k\to\infty} C_k = \infty$, and q was chosen so that $Q - q - \sigma < 0$. As $t_k \searrow 0$, it follows that $\lim_{k\to\infty} t_k^{Q-q-\sigma} = \infty$. If k is large enough that $t_k < \delta$, the monotonicity of I and the inequality (3.1.4) imply that

$$\int_{E_\delta} d(x,E)^{-\sigma} \, \mathrm{d}\mu(x) = I(\delta) \geq I(t_k) \geq C_k t_k^{Q-q-\sigma}.$$

The rightmost term is unbounded, implying that the integral diverges. □

In short, the integral defining the distance zeta gives a function on the real half-line to the right of the upper Minkowski dimension of E, and diverges on the complementary half-line. Lemma 3.6, which is a generalization of [LRŽ16, Lemma 2.19] to the current setting, extends the half-line of convergence to half-plane of convergence.

Lemma 3.6. *Let $E \subseteq X$ be bounded and fix $\delta > 0$. If $\zeta_E(s_0)$ converges (as a Lebesgue integral) for some $s_0 \in \mathbb{C}$, then $\zeta_E(s)$ converges for any $s \in \mathbb{C}$ such that $\Re(s) > \Re(s_0)$.*

Proof. The hypothesis that $\zeta_E(s_0)$ converges as a Lebesgue integral may be restated as

$$\int_{E_\delta} \left| d(x,E)^{s_0-Q} \right| < \infty.$$

Assume without loss of generality that $\delta < 1$, an assumption justified by Lemma 3.2. Note that if $a \in (0,1)$, then the real-valued function on $\mathbb{R}$ which takes σ to a^σ is decreasing. Hence, as $d(x,E) < \delta < 1$ for any $x \in E_\delta$,

$$\left| d(x,E)^{s-Q} \right| = d(x,E)^{\Re(s)-Q} \leq d(x,E)^{\Re(s_0)-Q} = \left| d(x,E)^{s_0-Q} \right|.$$

Integrating the above inequality gives the bound

$$\int_{E_\delta} \left|d(x,E)^{s-Q}\right| \, \mathrm{d}\mu(x) \leq \int_{E_\delta} \left|d(x,E)^{s_0-Q}\right| \, \mathrm{d}\mu(x) < \infty,$$

which finishes the proof. □

Remark 3.7. In summary, if $\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E)$, then the integral (3.1.1) will converge absolutely. On this half-plane, this *a priori* formal integral corresponds to a well-defined function, which gives a minimimal condition for what could be considered an "appropriate domain" in Definition 3.1. On the other hand, if $\Re(s) < \overline{\dim}_{\mathrm{Mi}}(E)$, this integral diverges. The upper Minkowski dimension of E marks the boundary of the largest right half-plane on which the distance zeta function may be explicitly defined as the integral (3.1.1).

Definition 3.8. Let $E \subseteq X$ be bounded and let $\delta > 0$. The *abscissa of* (*absolute*) *convergence* of the distance zeta function the real value

$$D_C(\zeta_E) := \inf \left\{ \sigma \in \mathbb{R} \;\middle|\; \int_{E_\delta} d(x,E)^{\sigma-Q} \, \mathrm{d}\mu(x) < \infty \right\}$$

In light of this definition and the preceding remark, $D_C(\zeta_E) = \overline{\dim}_{\mathrm{Mi}}(E)$.

3.2 The tube zeta function

Note that if $E \subseteq X$ is bounded and $t > 0$ is fixed, then the value of $d(x,E)$ is constant on the set

$$\{x \mid d(x,E) = t\}.$$

This suggests that a change of variables will allow the distance zeta function be evaluated over a real interval, rather than over a complex domain—this is, in essence, the idea of the "shell method" of integration introduced in elementary calculus classes. The computation is carried out in the following lemma.

Lemma 3.9. *If $E \subseteq X$ is bounded, then*

$$\int_{E_\delta} d(x,E)^{-\sigma}\,\mathrm{d}\mu(x) = \delta^{-\sigma}\mu(E_\delta) + \sigma\int_0^\delta t^{-\sigma-1}\mu(E_t)\,\mathrm{d}t < \infty \tag{3.2.1}$$

for every $\sigma < Q - \overline{\dim}_{\mathrm{Mi}}(E)$.

Proof. The Harvey-Polking estimate (Lemma 3.3) ensures that the integral on the left-hand side is finite, thus it remains to show that the two integrals are equal. The first step of the argument is to show that

$$\int_{E_\delta} d(x,E)^{-\sigma}\,\mathrm{d}\mu(x) = \int_0^\delta t^{-\sigma}\,\mathrm{d}V(t), \tag{3.2.2}$$

where $V(t)$ is the volume of a t-neighborhood of E, that is $V(t) = \mu(E_t)$ for all $t > 0$ and $V(0) = \mu(\overline{E})$. To prove the claim, define

$$T : X \to [0,\infty) : x \mapsto d(x,E)\chi_{E_\delta}(x), \qquad \text{and} \qquad f : \mathbb{R} \to [0,\infty] : t \mapsto t^{-\sigma}.$$

The generalized change of variables formula (Theorem 2.2) implies that

$$\int_X f\circ T(x)\,\mathrm{d}\mu(x) = \int_{[0,\infty]} f(t)\,\mathrm{d}T_*\mu(t). \tag{3.2.3}$$

Expand the left-hand side of (3.2.3) to obtain

$$\begin{aligned}\int_X f\circ T(x)\,\mathrm{d}\mu(x) &= \int_X \left[d(x,E)\chi_{E_\delta}(x)\right]^{-\sigma}\,\mathrm{d}\mu(x)\\ &= \int_{E_\delta} d(x,E)^{-\sigma}\,\mathrm{d}\mu(x),\end{aligned} \tag{3.2.4}$$

which is precisely the integral on the left-hand side of (3.2.2). Next, observe that

$$T_*\mu([0,t)) = \mu\left(T^{-1}\big([0,t)\big)\right) = \mu\big(\{x\in X \mid 0 \le d(x,E) < \min\{t,\delta\}\}\big)$$

for any $t \geq 0$, and so

$$T_*\mu([0,t)) = \begin{cases} \mu(\overline{E}) & \text{if } t = 0, \\ \mu(E_t) & \text{if } 0 < t < \delta, \text{ and} \\ \mu(E_\delta) & \text{if } t \geq \delta. \end{cases}$$

From this presentation, it is apparent that $T_*\mu([0,t))$ is the volume $V(t)$ of the t-neighborhood for any t between zero and δ. Make the substitution $\mathrm{d}T_*\mu(t) = \mathrm{d}V(t)$ in the right-hand side of (3.2.3) to get

$$\int_{[0,\infty]} f(t)\,\mathrm{d}T_* = \int_0^\delta t^{-\sigma}\,\mathrm{d}V(t). \tag{3.2.5}$$

With the substitutions given in (3.2.4) and (3.2.5), the identity at (3.2.3) is

$$\int_{E_\delta} d(x,E)^{-\sigma}\,\mathrm{d}\mu(x) = \int_0^\delta t^{-\sigma}\,\mathrm{d}V(t),$$

which establishes equality at (3.2.2).

Observe next that for any $\varepsilon \in (0,\delta)$, the functions $t \mapsto t^\sigma$ and V are continuous and of bounded variation. The integration by parts formula gives

$$\int_\varepsilon^\delta t^{-\sigma}\,\mathrm{d}V(t) = t^{-\sigma}V(t)\Big|_{t=\varepsilon}^\delta - \int_\varepsilon^\delta V(t)\left(-\sigma t^{-\sigma-1}\right)\,\mathrm{d}t$$

(see, for example, [Fol99, Thm. 3.36]). Taking the limit as ε decreases to zero gives

$$\int_0^\delta t^{-\sigma}\,\mathrm{d}V(t) = \lim_{\varepsilon \searrow 0}\left(t^{-\sigma}V(t)\Big|_{t=\varepsilon}^\delta + \sigma\int_\varepsilon^\delta t^{-\sigma-1}V(t)\,\mathrm{d}t\right). \tag{3.2.6}$$

Let $q \in (\overline{\dim}_{\mathrm{Mi}}(E), Q-\sigma)$ and note that such a choice of q is possible by the assumption that $\overline{\dim}_{\mathrm{Mi}}(E) < Q-\sigma$. By Defintion 2.7, the set E has zero q-dimensional upper Minkowski content for any $q > \overline{\dim}_{\mathrm{Mi}}(E)$, hence

$$0 = \overline{\mathfrak{M}}^q(E) = \limsup_{\varepsilon \searrow 0} \frac{\mu(E_\varepsilon)}{\varepsilon^{Q-q}} = \limsup_{\varepsilon \searrow 0} \frac{V(\varepsilon)}{\varepsilon^{Q-q}}.$$

There is therefore some constant $C > 0$ depending only on δ such that $V(\varepsilon) \leq C\varepsilon^{Q-q}$ for all $\varepsilon \in (0, \delta]$. After multiplication by a factor of $\varepsilon^{-\sigma}$, this gives

$$\frac{V(\varepsilon)}{\varepsilon^{\sigma}} < C\varepsilon^{Q-q-\sigma}$$

for all $\varepsilon \in (0, \delta]$. As q was chosen so that $q < Q - \sigma$, it follows that $0 < Q - q - \sigma$ and so the squeeze theorem implies that $C\varepsilon^{Q-q-\sigma} \searrow 0$ as $\varepsilon \searrow 0$. Hence

$$\lim_{\varepsilon \searrow 0} \left(t^{-\sigma} V(t) \Big|_{t=\varepsilon}^{\delta} \right) = \delta^{-\sigma} V(\delta) - \lim_{\varepsilon \searrow 0} \frac{V(\varepsilon)}{\varepsilon^{\sigma}} = \delta^{-\sigma} \mu(E_\delta).$$

Therefore (3.2.6) can be rewritten as

$$\begin{aligned} \int_0^{\delta} t^{-\sigma} \, \mathrm{d}V(t) &= \lim_{\varepsilon \searrow 0} \left(t^{-\sigma} V(t) \Big|_{t=\varepsilon}^{\delta} + \sigma \int_{\varepsilon}^{\delta} t^{-\sigma-1} V(t) \, \mathrm{d}t \right) \\ &= \delta^{-\sigma} \mu(E_\delta) + \lim_{\varepsilon \searrow 0} \left(\sigma \int_{\varepsilon}^{\delta} t^{-\sigma-1} \mu(E_t) \, \mathrm{d}t \right) \\ &= \delta^{-\sigma} \mu(E_\delta) + \sigma \int_0^{\delta} t^{-\sigma-1} \mu(E_t) \, \mathrm{d}t, \end{aligned}$$

where the finiteness of the original integral ensures that the integral in the last line converges to a finite value. Therefore (3.2.2) becomes

$$\int_{E_\delta} d(x, E)^{-\sigma} \, \mathrm{d}\mu(x) = \int_0^{\delta} t^{-\sigma} \, \mathrm{d}V(t) = \delta^{-\sigma} \mu(E_\delta) + \sigma \int_0^{\delta} t^{-\sigma-1} \mu(E_t) \, \mathrm{d}t,$$

which is the desired result. □

Lemma 3.9 is an analog of [LRŽ16, Lemma 2.1.4]. The proof techniques used here are based on the the alternative proof given in that text (see [LRŽ16, pp. 53–4]). However, that proof relies on a relation between the Hausdorff and Lebesgue measures on $\mathbb{R}^d$. Here, it is necessary to pass through the pushforward measure.

If σ in (3.2.1) is replaced by $Q - s$, the left hand integral becomes the distance zeta function (perhaps restricted to a right real half-line). On the right hand side of that identity, the term

$$\delta^{-\sigma}\mu(E_\delta) = \delta^{s-Q}\mu(E_\delta)$$

extends analytically to an entire function, while the remaining integral term has analytic properties nearly identical to those of the distance zeta function (the factor of $Q - s$ may, potentially, cancel a pole of the integral term). This motivates the introduction of the tube zeta function:

Definition 3.10. Let $E \subseteq X$ be bounded. The *tube zeta function* associated to E is the complex valued function $\tilde{\zeta}_E : U \to \mathbb{C}$ defined by the integral

$$\tilde{\zeta}_E(s) := \int_0^\delta t^{s-Q-1}\mu(E_t)\,\mathrm{d}t$$

where $\delta > 0$ and $U \subseteq \mathbb{C}$ is an appropriate domain.

Remark 3.11. After replacing $-\sigma$ with $s - Q$, the identity (3.2.1) becomes

$$\zeta_E(s) = \delta^{s-Q}\mu(E_\delta) + (Q - s)\tilde{\zeta}_E(s) < \infty$$

for all real $s > \overline{\dim}_{\mathrm{Mi}}(E)$.

3.3 Analyticity of the fractal zeta functions

The distance and tube zeta functions are examples of *fractal zeta functions*. In this section, the analytic properties of these fractal zeta functions are examined.

Definition 3.12. Let $U \subseteq \mathbb{C}$ and suppose that $F : U \to \mathbb{C}$. Further suppose that there is some $\tau \in [-\infty, \infty)$ such that F is holomorphic on the right half-plane $\{\Re(s) > \tau\} \subseteq U$. The *abscissa of holomorphic continuation* of F is the real value

$$D_H(F) := \inf\left\{\sigma \in \mathbb{R} \;\middle|\; F \text{ is holomorphic on } \{\Re(s) > \sigma\}\right\}.$$

Of particular interest will be $D_H(\zeta_E)$, i.e. the abscissa of convergence of the distance zeta function associated to a bounded subset E of X.

The major result of this section is given in Theorem 3.17, which demonstrates that, under relatively mild hypotheses on E, the abscissa of convergence $D_C(\zeta_E)$ and the abscissa of holomorphic continuation $D_H(\zeta_E)$ coincide. The proof of this result depends on several intermediate steps pertaining to Dirichlet type integrals (or DTIs), which are (roughly) functions of the form

$$F(s) := \int_X \varphi(x)^s \, \mathrm{d}\mu(x),$$

where φ is a suitable positive, μ-measurable function on X. The precise defintion and properties of DTIs are described in [LRŽ16, App. A].

The following two theorems are technical results regarding the analytic properties of DTIs. The statements are presented here, but the proofs are omitted. Proofs, as well as more discussion of these results, can be found in [LRŽ16].

Lemma 3.13. [LRŽ16, Thm. 2.1.45] *Let ν be a positive Radon measure on X and suppose that $\varphi : (X, \nu) \to \mathbb{R}_{>0}$ is a ν-measurable function. Further suppose there is some $C > 0$ such that $0 \leq \varphi(x) \leq C$ for ν-almost every $x \in X$ and that there is some $\sigma \in \mathbb{R}$ such that*

$$\int_X \varphi(x)^\sigma \, \mathrm{d}\nu(x) < \infty.$$

Then

$$F(s) := \int_X \varphi(x)^s \, \mathrm{d}\nu(x)$$

is holomorphic on the right half-plane $\{\Re(s) > \sigma\}$, and the derivative of F is given by

$$F'(s) = \int_X \varphi(x)^s \log(\varphi(x)) \, \mathrm{d}\nu(x)$$

in that region. Moreover,

$$D_H(F) \leq D_C(F).$$

That is, the abscissa of convergence gives an upper bound for the abscissa of holomorphic continuation.

A consequence of this theorem is an analog of [LRŽ16, Thm. 2.1.11(a)]. Of particular importance is the observation that the distance zeta function is holomorphic on the open half-plane to the right of the abscissa of convergence.

Corollary 3.14. *Let $E \subseteq X$ be bounded and let $\delta > 0$. The distance zeta function ζ_E is holomorphic on the open right half-plane $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E)\}$, and for all complex numbers s in that region, the derivative of the distance zeta function is given by*

$$\zeta_E'(s) = \int_{E_\delta} d(x,E)^{s-Q} \log(d(x,E)) \,\mathrm{d}\mu(x).$$

In addition,

$$D_H(\zeta_E) \leq D_C(\zeta_E).$$

Proof. Let ν be the Radon measure on X defined by

$$\mathrm{d}\nu(x) := d(x,E)^{-Q} \,\mathrm{d}\mu(x),$$

supported on $X \setminus E$. Define

$$\varphi(x) := d(x,E),$$

which is bounded by $C = \delta$ on the set E_δ. As per Remark 3.7,

$$\int_{E_\delta} \varphi(x)^\sigma \,\mathrm{d}\nu(x) = \int_{E_\delta \setminus E} d(x,E)^\sigma \, d(x,E)^{-Q} \,\mathrm{d}\mu(x) = \zeta_E(\sigma) < \infty.$$

The claimed results now follow immediately from Theorem 3.13. □

Corollary 3.15. *Let $E \subseteq X$ be bounded and let $\delta > 0$. The distance zeta function $\tilde{\zeta}_E$ is holomorphic on the open right half-plane $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E)\}$.*

Proof. Let ν be the Radon measure on $\mathbb{R}$ defined by

$$\mathrm{d}\nu(t) := \frac{1}{t^{Q+1}}\mu(E_t)\,\mathrm{d}t,$$

supported on $(0,\delta)$. Define

$$\varphi(t) = t,$$

which is bounded by $C = \delta$ on the interval $(0,\delta)$. It follows from Remark 3.11 that

$$\begin{aligned}\int_0^\delta \varphi(t)^\sigma\,\mathrm{d}\nu(t) = \int_0^\delta t\frac{1}{t^{Q+1}}\mu(E_t)\,\mathrm{d}t \\ = \tilde{\zeta}_E(s) = \frac{1}{Q-\sigma}\left(\zeta_E(s) - \delta^{\sigma-Q}\mu(E_\delta)\right) < \infty\end{aligned}$$

for any real $\sigma > \overline{\dim}_{\mathrm{Mi}}(E)$. The claimed result again follows from Theorem 3.13. □

Proposition 3.16. *Let $E \subseteq X$ be bounded, fix $\delta > 0$, and let $U \subseteq \mathbb{C}$ be a connected domain on which both the distance and tube zeta functions are holomorphic. Further suppose that*

$$U \cap \{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E)\} \neq \varnothing.$$

Then

$$\zeta_E(s) = \delta^{s-Q}\mu(E_\delta) + (Q-s)\tilde{\zeta}_E(s). \tag{3.3.1}$$

Proof. Without loss of generality, assume that $U \supseteq \{\Re(s) > D_H(\zeta_E)\}$. Corollaries 3.14 and 3.15 imply that both ζ_E and $\tilde{\zeta}_E$ are holomorphic on $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E)\}$. As per Remark 3.11, the identity (3.3.1) holds for $s \in (\overline{\dim}_{\mathrm{Mi}}(E), \infty)$. Therefore, by the Identity Theorem (see, for example, [Sim15, Thm. 2.3.8]), the identity holds throughout U. □

Theorem 3.17 is an important statement about the relation between the fractal zeta functions associated to E, and the geometry of that set—it gives sufficient hypotheses under which the fractal zeta functions will be singular at, and therefore "detect", the dimension of E. This theorem and its proof closely parallel [LRŽ16, Theorem 2.1.11(c)].

Theorem 3.17. *Let $E \subseteq X$ be bounded, let $\delta > 0$, and suppose that the Minkowski dimension of E exists. Further suppose that*

$$D := \dim_{\mathrm{Mi}}(E) < Q$$

and that $\underline{\mathfrak{M}}^D(E) > 0$. Then

$$\lim_{\sigma \searrow D} \zeta_E(\sigma) = +\infty,$$

where $\sigma \in \mathbb{R}$. Under these hypotheses, D is a singularity of ζ_E, and so $D_H(\zeta_E) = D_C(\zeta_E)$.

Proof. From the definition of the lower Minkowski content (Definition 2.7),

$$0 < \underline{\mathfrak{M}}^D(E) = \liminf_{t \searrow 0} \frac{\mu(E_t)}{t^{Q-D}},$$

from which it follows that for some sufficiently small δ, there exists a constant $C > 0$ such that

$$\frac{\mu(E_t)}{t^{Q-D}} > C \implies \mu(E_t) > C t^{Q-D} \tag{3.3.2}$$

whenever $t < \delta$. Then

$$\begin{aligned}
\lim_{\sigma \searrow D} \zeta_E(\sigma) &= \lim_{\sigma \searrow D} \left[\delta^{\sigma-Q} \mu(E_\delta) + (Q-\sigma) \tilde{\zeta}_E(\sigma) \right] && \text{(by Prop. 3.16)} \\
&\geq \lim_{\sigma \searrow D} \left[(Q-\sigma) \int_0^\delta t^{\sigma-Q-1} \mu(E_t) \, \mathrm{d}t \right] \\
&\geq \lim_{\sigma \searrow D} \left[(Q-\sigma) \int_0^\delta t^{\sigma-Q-1} \cdot C t^{Q-D} \, \mathrm{d}t \right] && \text{(by (3.3.2))} \\
&= C \lim_{\sigma \searrow D} \left[(Q-\sigma) \frac{t^{\sigma-D}}{\sigma - D} \right] \\
&= +\infty.
\end{aligned}$$

Hence ζ_E has a singularity at D, which implies that $D_H(\zeta_E) \geq D_C(\zeta_E)$. It then follows from Corollary 3.14 that

$$D_H(\zeta_E) = D_C(\zeta_E),$$

which is the desired result. □

Theorem 3.17 demonstrates that, under relatively mild hypotheses, the distance zeta function associated to a bounded set E will have a singularity at the real point corresponding to the upper Minkowski dimension of E. In this sense, the singularity of the distance zeta function is "a dimension" of E. By extension, every pole of an appropriately extended zeta function may be regarded as a complex valued dimension of the underlying set E. This is made formal in Definition 3.19, following the introduction of of some notation.

Definition 3.18. Let $f : U \to \mathbb{C}$ be a meromorphic function defined on some open domain $U \subseteq \mathbb{C}$. The set of *visible poles* of f (relative to U) defined to be

$$\mathscr{P}_U(f) := \{\omega \in U \mid \omega \text{ is a pole of } f\}.$$

Additionally, denote the collection of all poles of f by

$$\mathscr{P}(f) := \{\omega \in \mathbb{C} \mid \omega \in \mathscr{P}_U(f) \text{ for some } U\}.$$

Definition 3.19. Suppose that $E \subseteq X$ is a bounded set such that ζ_E admits a meromorphic extension to some open domain U containing the critical line $\{\Re(s) = D_C(\zeta_E)\}$. The sets

$$\mathscr{P}_U(\zeta_E) \qquad \text{and} \qquad \mathscr{P}(\zeta_E)$$

are the *visible complex dimensions* of E relative to U, and the *complex dimensions* of E, respectively.

3.4 Relative fractal drums

In the computation of the complex dimensions of the Sierpinski carpet in Example 1.1, one of the key observations is that several open squares are removed from the closed unit square at each stage of the construction and that each of these removed squares contributes to the distance zeta function in the same way, modulo a scaling factor. The main results of this section formalize this construction and computational technique.

Definition 3.20. Let $\Omega \subseteq X$ be an open set such that $\mu(\Omega) < \infty$, and let $E \subseteq X$ be arbitrary. Suppose that there is some $\delta > 0$ such that $\Omega \subseteq E_\delta$. The pair (E, Ω) is called a *relative fractal drum* (or RFD). Given an RFD (E, Ω), associate to it a *relative distance zeta function*

$$\zeta_{E,\Omega}(s) := \int_\Omega d(x, E)^{s-Q} \, \mathrm{d}\mu(x), \tag{3.4.1}$$

and a *relative tube zeta function*

$$\tilde{\zeta}_{E,\Omega}(s) := \int_0^\delta t^{s-Q-1} \mu(E_t \cap \Omega) \, \mathrm{d}t. \tag{3.4.2}$$

As suggested by the notation in Definition 3.1, if Ω is taken to be a δ-neighborhood of E, then

$$\zeta_{E,E_\delta} = \zeta_E \qquad \text{and} \qquad \tilde{\zeta}_{E,E_\delta} = \tilde{\zeta}_E.$$

Hence the distance and tube zeta functions can be viewed as special cases of relative distance and relative tube zeta functions, respectively.

As in the case of the zeta functions associated to bounded subsets of X, the integrals (3.4.1) and (3.4.2) generally only define analytic functions on an open right half-plane. The analysis is similar to that presented in Section 3.3, above. The basic definitions and results are outlined below.

Definition 3.21. Let (E, Ω) be an RFD in X, and let $q \in \mathbb{R}$—note that negative values of q are permissible. The *relative lower and upper q-dimensional Minkowski contents* of (E, Ω) are given by

$$\underline{\mathfrak{M}}^q(E, \Omega) := \liminf_{t \searrow 0} \frac{\mu(E_t \cap \Omega)}{t^{Q-q}} \quad \text{and} \quad \overline{\mathfrak{M}}^q(E, \Omega) := \limsup_{t \searrow 0} \frac{\mu(E_t \cap \Omega)}{t^{Q-q}},$$

respectively.

Definition 3.22. Let (E,Ω) be an RFD in X. The *relative lower and upper Minkowski dimensions* of (E,Ω) are given by

$$\begin{aligned}\underline{\dim}_{\mathrm{Mi}}(E,\Omega) &:= \inf\{q \in \mathbb{R} \mid \underline{\mathfrak{M}}^q(E,\Omega) = 0\} \\ &= \sup\{q \in \mathbb{R} \mid \underline{\mathfrak{M}}^q(E,\Omega) = +\infty\},\end{aligned}$$

and

$$\begin{aligned}\overline{\dim}_{\mathrm{Mi}}(E,\Omega) &:= \inf\{q \in \mathbb{R} \mid \overline{\mathfrak{M}}^q(E,\Omega) = 0\} \\ &= \sup\{q \in \mathbb{R} \mid \overline{\mathfrak{M}}^q(E,\Omega) = +\infty\},\end{aligned}$$

respectively. If both the relative lower and upper Minkowski dimensions are equal, then the common value is the *relative Minkowski dimension* of (E,Ω), denoted by $\dim_{\mathrm{Mi}}(E,\Omega)$.

Example 3.23. If E is bounded, and $\delta > 0$, then (for example)

$$\overline{\mathfrak{M}}^q(E,E_\delta) = \limsup_{t\searrow 0} \frac{\mu(E_t \cap E_\delta)}{t^{Q-q}} = \limsup_{t\searrow 0} \frac{\mu(E_t)}{t^{Q-q}} = \overline{\mathfrak{M}}^q(E),$$

where the second equality follows from noting that t is eventually smaller than δ. Hence relative Minkowski contents (and Minkowski dimensions) generalize the usual Minkowski contents (and dimensions, resp.) of bounded subsets of X. ◁

Example 3.24. Let (E,Ω) be an RFD in X, and suppose that $d(E,\Omega) = \varepsilon > 0$. Then for any $q \in \mathbb{R}$,

$$\overline{\mathfrak{M}}^r(E,\Omega) = \limsup_{t\searrow 0} \frac{\mu(E_t \cap \Omega)}{t^{Q-q}} = \limsup_{t\searrow 0} \frac{\mu(\varnothing)}{t^{Q-q}} = 0,$$

where the second equality follows from the observation that t is eventually smaller than ε. It then follows that

$$\overline{\dim}_{\mathrm{Mi}}(E,\Omega) = \inf\left\{q \in \mathbb{R} \;\middle|\; \overline{\mathfrak{M}}^q(E,\Omega) = 0\right\} = -\infty.$$

Thus, in contrast to the usual Minkowski dimensions, the relative Minkowski dimensions may take on negative values. ◁

Definition 3.25. Let (E, Ω) be an RFD in X. Let

$$D_C(\zeta_{E,\Omega}) := \inf \left\{ s_0 \in \mathbb{R} \;\middle|\; \int_\Omega \left| d(x,E)^{s-Q} \right| \, \mathrm{d}\mu(x) < \infty \text{ whenever } \Re(s) > s_0 \right\}.$$

That is, $D_C(\zeta_{E,\Omega})$ denotes the *abscissa of* (*absolute*) *convergence* of the relative zeta function $\zeta_{E,\Omega}$ (where this function is a Dirichlet-type integral).

Lemma 3.26. *Let (E, Ω) be an RFD in X. Then the integral defining the relative distance zeta function will converge on the open half-plane $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)\}$ and*

$$D_C(\zeta_{E,\Omega}) = \overline{\dim}_{\mathrm{Mi}}(E, \Omega).$$

Remark 3.27. Lemma 3.26 follows from an argument that is nearly identical to that used to establish the Harvey-Polking estimate in Lemma 3.3. As no new techniques are required here, the proof is omitted.

Theorem 3.28. *Let (E, Ω) be an RFD in X. On the open half-plane $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)\}$, the relative distance and tube zeta functions satisfy the functional equation*

$$\zeta_{E, E_\delta \cap \Omega}(s) = \delta^{s-Q} \mu(E_\delta \cap \Omega) + (Q - s) \tilde{\zeta}_{E,\Omega}(s).$$

Proof. By Theorem 3.26, the integral defining the relative distance tube zeta function converges on the half-plane $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)\}$. The remainder of the proof proceeds as in the proof of Lemma 3.9, replacing the measure μ with $\mu|_\Omega$, the restriction of μ to Ω. □

Theorem 3.29. *Let (E, Ω) be an RFD in X, suppose that $D := \dim_{\mathrm{Mi}}(E, \Omega) < Q$ exists, and $\overline{\mathfrak{M}}^D(E, \Omega) > 0$. Then the relative distance zeta function will be holomorphic on the half-plane*

$\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E,\Omega)$ *and singular at* D*. In this case,*

$$D_H(\zeta_{E,\Omega}) = D_C(\zeta_{E,\Omega}),$$

where D_H *is as in Defintion 3.12.*

Remark 3.30. The proof of Theorem 3.29 is nearly identical that of Theorem 3.17. As such, the proof is omitted.

Definition 3.31. Suppose that (E,Ω) is an RFD such that $\zeta_{E,\Omega}$ admits a meromorphic extension to some open domain U containing the critical line $\{\Re(s) = D_C(\zeta_{E,\Omega})\}$. The sets

$$\mathscr{P}_U(\zeta_{E,\Omega}) \qquad \text{and} \qquad \mathscr{P}(\zeta_{E,\Omega})$$

are called the *visible complex dimensions* of the RFD (E,Ω) relative to U, and the *complex dimensions* of the the RFD (E,Ω), respectively.

"Reasonable" notions of dimension defined for "reasonable" spaces are invariant with respect to similitudes (and, more generally, with respect to Lipschitz maps). As a rough example, if $E \subseteq \mathbb{R}^d$ and φ is a similitude on $\mathbb{R}^d$, then $\dim(\varphi(E)) = \dim(E)$, where dim is one of many common notions of dimension (e.g. the Hausdorff, Minkowski, or Assouad dimension). The complex dimensions of a relative fractal drum satisfy this "reasonableness" criterion, as outlined by the following theorem.

Theorem 3.32. *Let* (E,Ω) *be an RFD in* X*. Suppose that* $\varphi : X \to X$ *is a similitude with ratio* λ*, i.e.*

$$d(\varphi(x),\varphi(y)) = \lambda d(x,y)$$

for all $x, y \in X$*. Further suppose that*

$$\mu(\varphi(A)) = \lambda^Q \mu(A)$$

for all μ*-measurable sets* A (*for example, similarity maps on* $\mathbb{R}^d$ *with Lebesgue measure*)*. Then*

(*a*) $\zeta_{\varphi(E),\varphi(\Omega)}(s) = \lambda^s \zeta_{E,\Omega}(s)$ *for all* $\mathfrak{R}(s) > \overline{\dim}_{\mathrm{Mi}}(E,\Omega)$*, and*

(*b*) $\mathscr{P}(\zeta_{\varphi(E),\varphi(\Omega)}) = \mathscr{P}(\zeta_{E,\Omega})$.

Proof. Observe that for any measurable set $E \subseteq X$,

$$\left(\varphi^{-1}\right)_*(\mu)(E) = \mu\big((\varphi^{-1})^{-1}(E)\big) = \mu\left(\varphi(E)\right) = \lambda^Q \mu(E), \tag{3.4.3}$$

where the second equality follows from the fact that similarities are bijective. Applying the general change of variables formula (2.1.1),

$$\begin{aligned}
\zeta_{\varphi(E),\varphi(\Omega)}(s) &= \int_{\varphi(\Omega)} d(x,\varphi(E))^{s-Q}\,\mathrm{d}\mu(x) \\
&= \int_{\Omega} d(\varphi(y),\varphi(E))^{s-Q}\,\mathrm{d}\left(\varphi^{-1}\right)_*(\mu)(y) \\
&= \int_{\Omega} d(\varphi(y),\varphi(E))^{s-Q}\,\mathrm{d}\left(\lambda^Q\mu\right)(y) && \text{(by (3.4.3))} \\
&= \int_{\Omega} \lambda^{s-Q} d(y,E)^{s-Q}\lambda^Q\,\mathrm{d}\mu(y) && \text{(scaling property of } \varphi) \\
&= \lambda^s \int_{\Omega} d(y,E)^{s-Q}\,\mathrm{d}\mu(y) \\
&= \lambda^s \zeta_{E,\Omega}(s),
\end{aligned}$$

where the last integral converges for all $\mathfrak{R}(s) > \overline{\dim}_{\mathrm{Mi}}(E,\Omega)$. This establishes part (a) of the theorem. Part (b) then follows from from part (a) and the Identity Theorem for holomorphic functions. □

The main application of relative fractal drums in this thesis is in the computation of fractal zeta functions. The idea is to decompose a δ-neighborhood of a set E into a collection of RFDs, compute the relative zeta function corresponding to each such RFD, then "glue" everything back together again. This is made precise via the following definition and theorem, which are slightly generalized versions of [LRŽ16, Defn. 4.1.43 & Thm. 4.1.44].

Definition 3.33. Let (E_j,Ω_j) be a countable (finite or countably infinite) collection of RFDs in X such that

$$\Omega_j \cap \Omega_k = \varnothing$$

for all $j \neq k$ and

$$\mu\left(\bigcup_j \Omega_j\right) < \infty.$$

Then the *union of the family of RFDs* (E_j, Ω_j) is the RFD (E, Ω), where

$$E = \bigcup_j E_j \qquad \text{and} \qquad \Omega = \bigcup_j \Omega_j.$$

Let

$$\bigcup_j (E_j, \Omega_j) := (E, \Omega)$$

denote this union of RFDs.

Theorem 3.34. *Let (E_j, Ω_j) be a countable collection of RFDs satisfying the conditions of Definition 3.33, and let*

$$(E, \Omega) = \bigcup_j (E_j, \Omega_j).$$

Further suppose that if $x \in \Omega_j$ for some j, then

$$d(x, E) = d(x, E_j)$$

Then

$$\zeta_{E,\Omega}(s) = \sum_j \zeta_{E_j,\Omega_j}(s)$$

for any $s \in \mathbb{C}$ with $\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)$.

Proof. The proof follows along the same lines as that of [LRŽ16, Thm. 4.1.44]. If $\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)$, then the integral defining $\zeta_{E,\Omega}$ converges. For such s,

$$\begin{aligned}
\zeta_{E,\Omega}(s) &= \int_{\Omega} d(x,E)^{s-Q} \, \mathrm{d}\mu(x) \\
&= \sum_j \int_{\Omega_j} d(x,E)^{s-Q} \, \mathrm{d}\mu(x) && \text{(the } \Omega_j \text{ are disjoint)} \\
&= \sum_j \int_{\Omega_j} d(x,E_j)^{s-Q} \, \mathrm{d}\mu(x) && (x \in \Omega_j \implies d(x,\Omega) = d(x,\Omega_j)) \\
&= \sum_j \zeta_{E_j,\Omega_j}(s).
\end{aligned}$$

There is a small subtlety elided by this computation: it is not immediately clear that the integrals

$$\int_{\Omega_j} d(x,E_j)^{s-Q} \, \mathrm{d}\mu(x)$$

converge for all s with $\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)$. However, if σ is real (in particular, if $\sigma > \dim(E, \Omega)$), then $d(x, E_j)^{\sigma - Q}$ is nonnegative, and so

$$\int_{\Omega_j} d(x,E_j)^{\sigma-Q} \, \mathrm{d}\mu(x) \leq \sum_j \int_{\Omega_j} d(x,E_j)^{\sigma-Q} \, \mathrm{d}\mu(x) = \zeta_{E,\Omega}(\sigma).$$

Hence for real $\sigma > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)$, all of the relevant integrals converge. By an argument similar to that in Lemma 3.6, each one of these integrals converges on the half-plane $\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E, \Omega)\}$, as required. □

Chapter 4

Examples in p-adic Settings

4.1 The p-adic numbers

In this section, two standard constructions of the p-adic numbers are given: one from an analytic point of view, where the p-adic numbers are seen as a completion of the rationals with respect to the p-adic absolute value; and a second from an algebraic point of view, where the p-adic numbers are built from the projective limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$. The constructions are equivalent, but the distinct points of view make certain properties more readily apparent.

An analytic construction of $\mathbb{Q}_p$

The construction in this subsection parallels the construction presented by Gouvêa [Gou13, Ch. 3].

Definition 4.1. Let $\Bbbk$ be an arbitrary commutative ring with identity and $\mathbb{o}$ an arbitrary ordered field. Define the nonnegative part of $\mathbb{o}$ by $\mathbb{o}_+ := \{x \in \mathbb{o} \mid x \geq 0_{\mathbb{o}}\}$, where $0_{\mathbb{o}}$ is the additive identity element of $\mathbb{o}$. An *absolute value* on $\Bbbk$ is a function

$$|\cdot| : \Bbbk \to \mathbb{o}_+$$

which satisfies the following three axioms:

(A1) $|x| = 0_{\mathbb{o}}$ if and only if $x = 0_{\mathbb{o}}$ (i.e. $|\cdot|$ is definite);

(A2) $|xy| = |x||y|$ for all $x, y \in \Bbbk$; and

(A3) $|x + y| \leq |x| + |y|$ for all $x, y \in \Bbbk$ (i.e. $|\cdot|$ satisfies the triangle inequality).

The absolute value $|\cdot|$ is said to be *non-archimedean* if it satisfies a stronger version of (A3), namely the axiom

(A4) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \Bbbk$ (i.e. $|\cdot|$ satisfies the ultrametric inequality).

Otherwise, $|\cdot|$ is said to be *archimedean*.

Definition 4.2. Let p be a fixed prime number. Define a function $v_p : \mathbb{Q} \to \mathbb{Z}$ by $v_p(x) := n$, where n is the unique integer such that x can be written as

$$x = p^{v_p(x)} \frac{a}{b}$$

with $a, b \in \mathbb{Z}$ and $p \nmid a, b$. The function v_p is called the *p-adic valuation* on $\mathbb{Q}$.

Lemma 4.3 (see [Gou13, Prop. 2.1.5])**.** *The map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_+$ defined by $|x|_p = p^{-v_p(x)}$ is a non-archimedean absolute value on $\mathbb{Q}$. This absolute value is called the* p-adic absolute value *on $\mathbb{Q}$.*

The p-adic absolute value induces a metric on $\mathbb{Q}$, defined by $d_p(x, y) := |x - y|_p$. This metric in turn defines a topology, making $\mathbb{Q}$ into a metric space—indeed, an "ultrametric space," i.e. a metric space in which axiom (A4) is satisfied by the metric. However, $\mathbb{Q}$ is not complete with respect to any such metric. Thus the next goal is to complete $\mathbb{Q}$ in the usual manner, as outlined below for the sake of completeness.

Lemma 4.4 (see [Gou13, Prop. 3.2.5])**.** *Let C_p denote the collection of all sequences in $\mathbb{Q}$ that are Cauchy with respect to $|\cdot|_p$. With the operations*

$$(x_n) + (y_n) := (x_n + y_n), \qquad \textit{and} \qquad (x_n) \cdot (y_n) := (x_n y_n),$$

the set C_p is a commutative ring with identity.

Lemma 4.5 (see [Gou13, Lemma 3.2.8])**.** *Let $\mathcal{N}_p$ denote the null sequences in C_p. That is,*

$$\mathcal{N}_p := \left\{ (x_n) \in C_p \;\middle|\; \lim_{n\to\infty} |x_n|_p = 0 \right\}.$$

Then $\mathcal{N}_p$ is a maximal ideal in C_p.

Definition 4.6. The field of *p-adic numbers*, denoted $\mathbb{Q}_p$ is given by

$$\mathbb{Q}_p := C_p/\mathcal{N}_p.$$

Given $x \in \mathbb{Q}$, the constant sequence $(x) := (x, x, \dots) \in C_p$. Thus the map $x \mapsto (x)$ is an inclusion of $\mathbb{Q}$ into C_p. Since two distinct constant sequences differ by a nonzero constant sequence, this inclusion passes to the quotient. That is, there is an inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ via the map sending $x \in \mathbb{Q}$ to the constant sequence (x). This, combined with the following rather striking lemma, provides a way of uniquely extending the p-adic absolute value to $\mathbb{Q}_p$.

Lemma 4.7 (see [Gou13, Lemma 3.2.10])**.** *Let $(x_n) \in C_p \setminus \mathcal{N}_p$. Then there is some $N \in \mathbb{N}$ such that $|x_m|_p = |x_n|_p$ for all $m, n \geq N$. That is, the sequence $(|x_n|_p)$ is eventually stationary.*

This implies that if $(x_n) \in \mathbb{Q}_p$, then $\lim_{n\to\infty} |x_n|_p$ must be a finite real number. The p-adic absolute value on $\mathbb{Q}$ can be extended to an absolute value on $\mathbb{Q}_p$ in a straightforward manner.

Definition 4.8. If $x \in \mathbb{Q}_p$ and (x_n) is any Cauchy sequence in the equivalence class of x, define the *p-adic absolute value* on $\mathbb{Q}_p$ by

$$|x|_p := \lim_{n\to\infty} |x_n|_p .$$

It can be verified that this definition does not depend on representatives, and gives a non-archimedean absolute value on $\mathbb{Q}_p$ that is consistent with the p-adic absolute value on $\mathbb{Q}$—that is, $|x|_p = |(x)|_p$ for all $x \in \mathbb{Q}$. Finally, $\mathbb{Q}_p$ is Cauchy complete with respect to the p-adic absolute value, and the embedded image of $\mathbb{Q}$ is dense in $\mathbb{Q}_p$. Hence $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. This is made formal by Theorem 4.9.

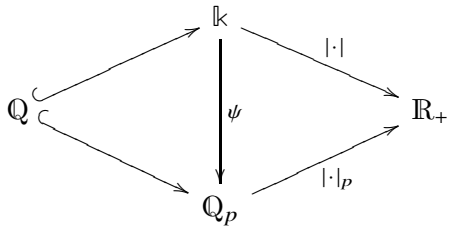
Theorem 4.9 (see [Gou13, Prop. 3.2.13])**.** *Suppose that $\Bbbk$ is a field with an associated non-archimedean absolute value $|\cdot|$. Further suppose that*

(*a*) *there exists an inclusion $\iota : \mathbb{Q} \hookrightarrow \Bbbk$ such that $|\iota(x)| = |x|_p$ for all $x \in \mathbb{Q}$;*

(*b*) *the set $\iota(\mathbb{Q})$ is dense in $\Bbbk$ (with respect to the topology induced by $|\cdot|$); and*

(*c*) *$\Bbbk$ is complete with respect to $|\cdot|$.*

Then there is a unique isomorphism ψ such that the diagram



commutes.

An algebraic construction of $\mathbb{Q}_p$

This subsection follows the presentations of Robert [Rob13, Ch. 1] and Serre [Ser12, Ch. II].

Definition 4.10. Let p be a fixed prime number. For each $j, k \in \mathbb{N}$ with $j \leq k$, let

$$\pi_j^k : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}$$

be the natural projection map (i.e. $\pi_j^k(a) = a \pmod{p^j}$). The *p-adic integers* are the set

$$\mathbb{Z}_p := \left\{ (a_1, a_2, \ldots) \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} \;\middle|\; \pi_j^k(a_k) = a_j \forall j, k \in \mathbb{N} \right\}.$$

Observe that if $(a_n) \in \mathbb{Z}_p$, then the requirement that $\pi_{k-1}^k(a_k) = a_{k-1}$ implies that $a_k = a_{k-1} + \alpha_k p^{k-1}$ for some $\alpha_k \in \{0, 1, \ldots, p-1\}$. Thus an element of $\mathbb{Z}_p$ can be written as

$$(a_n) = \left(\alpha_1, \alpha_1 + \alpha_2 p, \alpha_1 + \alpha_2 p + \alpha_3 p^2, \ldots, \sum_{k=1}^{n} \alpha_k p^{k-1}, \ldots \right)$$

where $\alpha_k \in \{0, 1, \ldots, p-1\}$ for each $k \in \mathbb{N}$. Hence if $(a_n) \in \mathbb{Z}_p$, it has a unique series representation of the form

$$(a_n) = \sum_{k=1}^{\infty} \alpha_k p^{k-1}, \qquad \text{where } \alpha_k \in \{0, 1, \ldots, p-1\} \text{ for all } k. \tag{4.1.1}$$

Therefore the elements of $\mathbb{Z}_p$ may be viewed either as sequences of integers modulo powers of p that are compatible with the natural projection maps, or as formal power series in p with coefficients in the set $\{0, 1, \ldots, p-1\}$. In the following discussion, whichever representation makes the exposition more transparent will be used.

If $a \in \mathbb{Z}$, then the map $a \mapsto (a \pmod{p^n})$ gives a natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$. Via this inclusion, $\mathbb{Z}$ is reasonably viewed as a subset of $\mathbb{Z}_p$ (specifically, the subset of $\mathbb{Z}_p$ possessing series representations with only finitely many nonzero terms; or the subset of $\mathbb{Z}_p$ having an eventually constant sequence representation).

While $\mathbb{Z}_p$ is *a priori* constructed only as a set, it has a natural ring structure, with addition and multiplication performed termwise with respect to the sequence representation. That is, if $(a_n), (b_n)$ are sequences in $\mathbb{Z}_p$, then

$$(a_n) + (b_n) := (a_n + b_n) \qquad \text{and} \qquad (a_n) \cdot (b_n) := (a_n \cdot b_n),$$

where the termwise addition and multiplication take place in the rings $\mathbb{Z}/p^n\mathbb{Z}$.

Theorem 4.11. *The ring of p-adic integers is a principal ideal domain. Moreover, every ideal in $\mathbb{Z}_p$ is of the form $p^n\mathbb{Z}_p$ for some $n \in \mathbb{N} \cup \{\infty\}$ (where, in order to simplify notation,, define $p^\infty\mathbb{Z}_p = \{0\}$).*

The next task is to topologize $\mathbb{Z}_p$. For each $k \in \mathbb{N}$ take $\varphi_k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$ to be the projection homomorphism given by

$$\varphi_k\left((a_1, a_2, a_3, \dots)\right) = a_k.$$

A basis for a topology on $\mathbb{Z}_p$ is given by sets of the form $\varphi_k^{-1}(a)$, where $k \in \mathbb{N}$ and $a \in \mathbb{Z}/p^k\mathbb{Z}$. The topology generated by this basis is called the *profinite topology*. It is worth noting that the ideals in $\mathbb{Z}_p$ are the basis open sets $\varphi_k^{-1}(1)$.

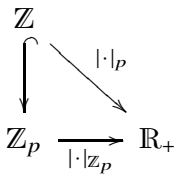
Theorem 4.12. *The ring of p-adic integers under the profinite topology is compact, Hausdorff, and totally disconnected. Moreover, the image of* $\mathbb{Z}$ (*via the natural inclusion*) *is dense in* $\mathbb{Z}_p$.

Definition 4.13. Let $a \in \mathbb{Z}_p$, and let n be the unique positive integer (or ∞) such that $\langle a \rangle$, the ideal generated by a, is $p^n\mathbb{Z}_p$. Define a map $|\cdot|_{\mathbb{Z}_p} : \mathbb{Z}_p \to \mathbb{R}_+$ by setting

$$|a|_{\mathbb{Z}_p} := p^{-n}.$$

Proposition 4.14. *The map* $|\cdot|_{\mathbb{Z}_p}$ *is a non-archimedean absolute value on* $\mathbb{Z}_p$.

Not only is $|\cdot|_{\mathbb{Z}_p}$ a non-archimedean absolute value on $\mathbb{Z}_p$, it agrees with the restriction of the p-adic absolute value to $\mathbb{Z}$. That is, the diagram



commutes. Moreover, the absolute value on $\mathbb{Z}_p$ is complete (in the sense that the induced metric is Cauchy complete).

As the p-adic integers are a principle ideal domain, they possess an associated field of fractions. Let $\widehat{\mathbb{Z}}_p$ denote this field of fractions. There is a unique inclusion $\iota : \mathbb{Q} \hookrightarrow \widehat{\mathbb{Z}}_p$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \iota \\ \mathbb{Z}_p & \hookrightarrow & \widehat{\mathbb{Z}}_p \end{array}$$

commutes, where the horizontal maps are the canonical embeddings of the respective rings into their associated fields of fractions, and the map on the left is the natural embedding of $\mathbb{Z}$ into $\mathbb{Z}_p$. A basis for a topology on $\widehat{\mathbb{Z}}_p$ is given by sets of the form $a + p^n\mathbb{Z}_p$, where $a \in \iota(\mathbb{Q})$ and $n \in \mathbb{Z}$.

Theorem 4.15. *Under the above defined topology, $\widehat{\mathbb{Z}}_p$ is a locally compact topological field. The ring of p-adic integers, $\mathbb{Z}_p$, is its maximal compact subring.*

By construction, the basis is countable. Moreover, for any $a \in \iota(\mathbb{Q})$ and $n \in \mathbb{Z}$,

$$a + p^{n-1}\mathbb{Z}_p = a + p^n\mathbb{Z}_p \sqcup \bigsqcup_{k=1}^{p-1} b_{1,k} + p^n\mathbb{Z}_p$$

for some points $b_{1,k} \in \iota(\mathbb{Q})$. Similarly, for any $j \in \mathbb{N}$, it is possible to write $a + p^{n-j}\mathbb{Z}_p$ as the disjoint union of p sets of the form $b_{j,k} + p^{n-j+1}\mathbb{Z}_p$ (where $b_{0,j} = a$). Hence, by a recursive construction,

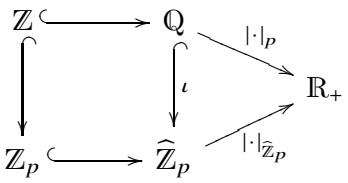
$$\widehat{\mathbb{Z}}_p \setminus a + p^n\mathbb{Z}_p = \bigsqcup_{j=1}^{\infty} \bigsqcup_{k=1}^{p-1} b_{j,k} + p^{n-j+1}\mathbb{Z}_p.$$

But this is a countable union of open sets, and is therefore open. Therefore the basis open sets are also closed. It follows that $\widehat{\mathbb{Z}}_p$ is regular, and so by Urysohn's lemma, the space $\widehat{\mathbb{Z}}_p$ is metrizable.

Rather than directly constructing a metric, construct an absolute value on $\widehat{\mathbb{Z}}_p$ which induces a metric. To wit, define

$$\left|\frac{a}{b}\right|_{\widehat{\mathbb{Z}}_p} := \frac{|a|_{\mathbb{Z}_p}}{|b|_{\mathbb{Z}_p}}.$$

This map extends the absolute value on $\mathbb{Z}_p$ to a non-archimedean absolute value on $\widehat{\mathbb{Z}}_p$. Indeed, it follows from axiom A2 that this is the only possible extension. Then the diagram



commutes. Observe that balls of the form

$$B(x,r) = \left\{ y \in \widehat{\mathbb{Z}}_p \;\middle|\; |x-y| < r \right\}$$

are precisely the basis open sets for the topology on $\widehat{\mathbb{Z}}_p$. Indeed, $\mathbb{Z}_p$ is exactly the unit ball in $\mathbb{Q}_p$.

To summarize, $\widehat{\mathbb{Z}}_p$ is a topological field with a non-archimedean absolute value $|\cdot|_{\widehat{\mathbb{Z}}_p}$. The rational numbers embed into $\widehat{\mathbb{Z}}_p$ as a dense subset, and the absolute value in $\widehat{\mathbb{Z}}_p$ agrees with the p-adic absolute value on the image of $\mathbb{Q}$. This section concludes with a final result:

Lemma 4.16. *The field $\widehat{\mathbb{Z}}_p$ is complete with respect to $|\cdot|_{\widehat{\mathbb{Z}}_p}$.*

By invoking theorem 4.9,

$$\left(\widehat{\mathbb{Z}}_p, |\cdot|_{\widehat{\mathbb{Z}}_p}\right) \cong \left(\mathbb{Q}_p, |\cdot|_p\right),$$

where the isomorphism is uniquely determined. It is therefore reasonable to say that $\widehat{\mathbb{Z}}_p$ *is* $\mathbb{Q}_p$, and to view elements of $\mathbb{Q}_p$ in light of either the analytic construction of the previous subsection, or the algebraic construction of the current subsection.

Key properties of $\mathbb{Q}_p$

To fix notation, the space $(\mathbb{Q}_p, d_p)$ will denote the p-adic numbers taken together with the metric d_p, which is the metric induced by the p-adic absolute value. That is,

$$d_p(x,y) := |x-y|_p$$

for any $x, y \in \mathbb{Q}_p$.

In $\mathbb{R}$, open and closed intervals play a vital role analysis. The natural analog of an interval in $\mathbb{Q}_p$ is a ball. However, all balls in $\mathbb{Q}_p$ are clopen, thus open and closed balls don't quite play the same role as open and closed intervals, respectively. Instead, the appropriate analogs are stripped and dressed balls.

Definition 4.17. Let $x \in \mathbb{Q}_p$ and $r > 0$. The *stripped ball* of radius r centered at x is the set

$$B_<(x, r) := \left\{ y \in \mathbb{Q}_p \;\middle|\; d_p(x, y) < r \right\},$$

and the *dressed ball* of radius r centered at x is the set

$$B_\leq(x, r) := \left\{ y \in \mathbb{Q}_p \;\middle|\; d_p(x, y) \leq r \right\}.$$

Stripped and dressed balls generally play the role in $\mathbb{Q}_p$ that is played by open and closed intervals in $\mathbb{R}$.

The following theorem states several of the key properties of the p-adic numbers. The somewhat idiosyncratic descriptions of these properties are inspired by Gouvêa [Gou13].

Theorem 4.18. *Let $x, y, z \in \mathbb{Q}_p$ be arbitrary, and fix $r, s > 0$.*

(*a*) (*The biggest wins.*) *If $|x|_p \neq |y|_p$ then*

$$|x + y|_p = \max\{|x|_p, |y|_p\}.$$

(*b*) (*All triangles are isosceles.*) *At least two of*

$$d_p(x, y), \quad d_p(x, z), \quad \textit{and} \quad d_p(y, z)$$

are equal.

(*c*) (*Every point is the center.*) *If* $a \in B_{\leq}(x,r)$*, then* $B_{\leq}(a,r) = B_{\leq}(x,r)$*. The same result holds for the stripped ball.*

(*d*) (*Venn diagrams are boring.*) *If* $B_{\leq}(x,r) \cap B_{\leq}(y,s) \neq \emptyset$*, then either*

$$B_{\leq}(x,r) \subseteq B_{\leq}(y,s) \quad \textit{or} \quad B_{\leq}(y,s) \subseteq B_{\leq}(x,r).$$

The same result holds for the stripped balls.

It follows from Theorem 4.18 that a ball of radius p^n in $\mathbb{Q}_p$ is the disjoint union of p balls of radius p^{n-1}. Specifically, if $x \in \mathbb{Q}_p$, $n \in \mathbb{Z}$, and $b \in \{0, 1, \ldots, p-1\}$, then

$$B_{\leq}(x, p^n) = \bigsqcup_{b=0}^{p-1} B_{\leq}(x + b, p^{n-1}). \tag{4.1.2}$$

Because the notation will be convenient later, define

$$pE + b := \{px + b : x \in E\},$$

where $E \subseteq \mathbb{Q}_p$ and $b \in \{0, 1, \ldots, p-1\}$. In this notation, (4.1.2) can be rewritten as

$$B_{\leq}(x, p^n) = \bigsqcup_{b=0}^{p-1} pB_{\leq}(x, p^n) + b.$$

For example, in $\mathbb{Q}_7$, the dressed unit ball $\mathbb{Z}_7$ can be understood as in Figure 4.1. Note that in this setting multiplication by p is contractive. Indeed, multiplication by p is a contracting similitude with

$$d_p(px, py) = p^{-1} d_p(x, y)$$

for all $x, y \in \mathbb{Q}_p$.

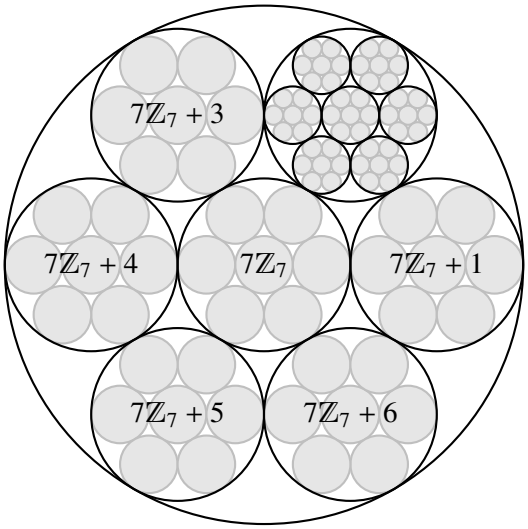


Figure 4.1: A diagram of the 7-adic integers. The unit ball, $\mathbb{Z}_7$, is composed of seven disjoint balls of radius $\frac{1}{7}$ (namely, $7\mathbb{Z}_7 + j$ for $j \in \{0, 1, \ldots, 6\}$). Each of these is, in turn, composed of seven disjoint balls of radius $\frac{1}{49}$, and so on.

The p-adic numbers are endowed with a natural Haar measure μ_p, which is normalized so that the p-adic integers have measure 1. Under this normalization,

$$\mu_p(B(x, p^{-k})) = p^{-k}$$

for any $x \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$.

Proposition 4.19. *For any prime number p,*

$$\dim_{\mathrm{As}}(\mathbb{Q}_p) = 1.$$

Proof. Fix $0 < \rho < r$ and let $x \in \mathbb{Q}_p$ be arbitrary. Let $\kappa, k \in \mathbb{Z}$ be the uniqe integers such that

$$p^{-\kappa} \leq \rho < p^{-(\kappa-1)} \qquad \text{and} \qquad p^{-(k+1)} < r \leq p^{-k}.$$

Then

$$B(x, r) = B(x, p^{-k}) \qquad \text{and} \qquad B(\xi, \rho) = B(\xi, p^{-(\kappa-1)}) \supseteq B(\xi, p^{-\kappa})$$

for any $\xi \in B(x,r)$. For any such ξ, it then follows that

$$\frac{\mu_p(B(x,r))}{\mu_p(B(\xi,\rho))} = \frac{\mu_p(B(x,p^{-k}))}{\mu_p(\xi,p^{-(\kappa-1)})} = \frac{p^{-k}}{p^{-(\kappa-1)}} = p\left(\frac{p^{-k}}{p^{-\kappa}}\right) \leq p\left(\frac{r}{\rho}\right)^1 .$$

Therefore μ_p is 1-homogeneous on $\mathbb{Q}_p$. Fix some $q \in [0,1)$, and suppose for contradiction that μ_p is q-homogeneous on $\mathbb{Q}_p$. That is, suppose that there is some $M > 0$ such that

$$\frac{\mu_p(B(x,r))}{r^q} \leq M\left(\frac{\mu_p(B(\xi,\rho))}{\rho^q}\right)^q$$

for all $0 < \rho < r$, all $x \in \mathbb{Q}_p$, and all $\xi \in B(x,r)$. With $r = p^{-k}$ and $\rho = p^{-\kappa}$ for some $\kappa > k \in \mathbb{Z}$, this implies that

$$\frac{p^{-k}}{p^{-\kappa}} \leq M\left(\frac{p^{-k}}{p^{-\kappa}}\right)^q \iff p^{(1-q)(\kappa-k)} \leq M$$

for all $\kappa > k \in \mathbb{Z}$. But $p^{(1-q)(\kappa-k)}$ can be made arbitrarily large by taking κ to be enough larger than k. Hence no such M can exist, contradicting the assumption that μ_p is q-homogeneous for some $q \in [0,1)$. Therefore

$$\dim_{\mathrm{As}}(\mathbb{Q}_p) = \inf\{q \mid \mu_p \text{ is } q\text{-homogeneous on } \mathbb{Q}_p\} = 1,$$

which is the claimed result. □

Vector spaces over $\mathbb{Q}_p$

Let p be a fixed prime number and $Q \in \mathbb{N}$, and let $\mathbb{Q}_p^Q$ denote the Q-dimensional vector space over $\mathbb{Q}_p$ For any $\alpha \in [1,\infty)$, define a metric on the product space by setting

$$d_p^\alpha(\boldsymbol{x},\boldsymbol{y}) := \left(\sum_{k=1}^N |x_k - y_k|^\alpha\right)^{1/\alpha} .$$

The L_∞-metric (or the max-metric) is defined by

$$d_p^\infty(\boldsymbol{x},\boldsymbol{y}) := \max_{k\leq N}\{|x_k - y_k|\} .$$

Note that for any $\alpha, \beta \in [1, \infty]$, the metrics d_p^α and d_p^β induce the same topology, and are therefore equivalent as metrics. However, only the d_p^∞-metric possesses the non-archimedean property, and may, in some sense, be regarded as the natural metric on $\mathbb{Q}_p^Q$. In light of this, write

$$d_p := d_p^\infty.$$

Such vector spaces over $\mathbb{Q}_p$, endowed with the natural product measure, are homogeneous and satisfy

$$\dim_{\mathrm{As}}(\mathbb{Q}_p^Q) = Q.$$

4.2 Examples in $\mathbb{Q}_p$

Singleton sets in $\mathbb{Q}_p$

This section begins with a computation of the complex dimensions of a singleton set in $\mathbb{Q}_p$, where $\mathbb{Q}_p$ is equipped with its natural Haar measure μ_p, normalized so that $\mu_p(\mathbb{Z}_p) = 1$.

Example 4.20. It follows from the translation invariance of the measure on $\mathbb{Q}_p$ that any singleton set will have the same complex dimensions as any other singleton set. Thus for notationally simplicity, it suffices to compute the complex dimensions of the singleton set $\{0\}$. Then, for s with sufficiently large real part,

$$\begin{aligned}
\zeta_{\{0\}}(s) &= \int_{\mathbb{Z}_p} d(x,0)^{s-1} \, \mathrm{d}\mu_p(x) \\
&= \sum_{n=0}^{\infty} \int_{p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p} p^{-n(s-1)} \, \mathrm{d}\mu_p(x) \\
&= \sum_{n=0}^{\infty} p^{n(1-s)} \left[p^{-n} - p^{-n-1} \right] \\
&= \frac{p-1}{p} \sum_{n=0} \left(\frac{1}{p^s} \right)^n \\
&= \frac{p-1}{p} \frac{p^s}{p^s - 1}.
\end{aligned}$$

Hence $\zeta_{\{0\}}$ can be extended to a mentire function with simple poles occurring whenever $p^s = 1$. That is

$$\mathscr{P}(\zeta_{\{0\}}) = 0 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}.$$

This example demonstrates the somewhat surprising result that singleton sets in $\mathbb{Q}_p$ possess geometric oscillation. This kind of highly localized oscillatory behaviour is explored in greater detail in Chapter 5. ◁

For comparison, the corresponding computation in $\mathbb{R}$ exhibits markedly different behaviour. Let m denote the usual Lebesgue measure. This measure is translation invariant, thus $\zeta_{\{x\}} = \zeta_{\{0\}}$ for any $x \in \mathbb{R}$. The distance zeta function corresponding to a singleton point in $\mathbb{R}$ is

$$\zeta_{\{0\}}(s) = \int_{(-1,1)} d(x,0)^{s-1}\,\mathrm{d}m(x) = 2\int_0^1 x^{s-1}\,\mathrm{d}x = \frac{2}{s}.$$

This zeta function can be extended to a mentire function which possesses a simple pole at $s = 0$ (and no other poles or singularities). In contrast to points in $\mathbb{Q}_p$, points in $\mathbb{R}$ do not exhibit oscillatory geometric behaviour.

Balls in $\mathbb{Q}_p$

Example 4.21. Any ball in $\mathbb{Q}_p$ may be sent to the unit ball $\mathbb{Z}_p$ via a similarity, and μ_p is measure scaling in the sense of Theorem 3.32. Hence

$$\zeta_{B_{\leq}(x,r)} = k\zeta_{\mathbb{Z}_p}$$

for any $x \in \mathbb{Q}_p$ and any $r > 0$, where k is a constant depending only on r. With $\delta < 1$, the a δ-neighborhood of $\mathbb{Z}_p$ is simply $\mathbb{Z}_p$, and so

$$\zeta_{\mathbb{Z}_p}(s) = \int_{\mathbb{Z}_p} d(x,\mathbb{Z}_p)^{s-1}\,\mathrm{d}\mu_p(x) = \int_{\mathbb{Z}_p} 0^{s-1}\,\mathrm{d}\mu_p(x) = 0$$

for all $s > 1$. This extends analytically to the zero function on $\mathbb{C}$, hence the unit ball $\mathbb{Z}_p$ has no complex dimensions. In particular, note that $\dim_{\mathrm{Mi}}(\mathbb{Z}_p) = 1$ is not a complex dimension of $\mathbb{Z}_p$. This is not surprising, as $\mathbb{Z}_p$ fails to satisfy the hypotheses of Theorem 3.17—specifically,

$$\overline{\dim}_{\mathrm{Mi}}(\mathbb{Z}_p) = 1 \not< 1 = \dim_{\mathrm{As}}(\mathbb{Q}_p).$$

From the point of view of the fractal zeta functions, the unit ball in $\mathbb{Q}_p$ has little observable geometric structure. ◁

By contrast, the distance zeta function does see some of the geometry of the unit ball (the interval $(-1, 1)$) in $\mathbb{R}$. With $\delta < 1$ and s chosen with sufficiently large real part, this zeta function is given by

$$\zeta(-1,1)(s) = \int_{-1-\delta}^{1+\delta} d(x,0)^{s-1}\,\mathrm{d}m(x) = 2\int_{1}^{1+\delta} x^{s-1}\,\mathrm{d}x = \frac{2(1+\delta)^2}{s} + \frac{2}{s}.$$

This can be extended to a mentire function with a simple pole at $s = 0$. The zeta function fails to detect the Minkowski dimension of the interval, though this is again unsurprising, as the interval fails to satisfy the hypotheses of Theorem 3.17. However, in this real case, the distance zeta function still sees some geometry—the pole at zero corresponds to the zero dimensional boundary (the points ± 1) of the interval.

A self-similar measure on $\mathbb{Z}_2$

Example 4.22. Let $\{\varphi_i : \mathbb{Q}_2 \to \mathbb{Q}_2\}_{i=0}^{1}$ be the IFS with maps

$$\varphi_0(x) = 2x \qquad \text{and} \qquad \varphi_1(x) = 2x + 1.$$

Take $\mu_{\mathfrak{p}}$ to be the self-similar measure on $\mathbb{Z}_2$ corresponding to this system with the weights

$$\mathfrak{p}_0 = \frac{1}{3} \qquad \text{and} \qquad \mathfrak{p}_1 = \frac{2}{3}.$$

To determine the ambient dimension of this space, take $r = 2^{-m}$ and $\rho = 2^{-m-n}$ where $m, n \in \mathbb{N}$ are arbitrary, and suppose that $x \in \mathbb{Z}_2$ and $\xi \in B(x, r)$. There exist $\boldsymbol{i}, \boldsymbol{j} \in \{0, 1\}^*$ such that

$$B(x, r) = \varphi_{\boldsymbol{i}}(\mathbb{Z}_2) \qquad \text{and} \qquad B(\xi, \rho) = \varphi_{\boldsymbol{i}} \circ \varphi_{\boldsymbol{j}}(\mathbb{Z}_2).$$

Then

$$\left(\frac{1}{3}\right)^m \leq \mu(B(x, r)) = \mathfrak{p}_{\boldsymbol{i}} \leq \left(\frac{2}{3}\right)^m,$$

with equality on the left for $\boldsymbol{i} = (0, 0, \ldots, 0)$ and equality on the right for $\boldsymbol{i} = (1, 1, \ldots, 1)$. It is worth noting that this implies that $(\mathbb{Z}_2, \mu_{\mathfrak{p}})$ is *not* Ahlfors regular. In then follows that

$$\mu_{\mathfrak{p}}(B(\xi, \rho)) = \mathfrak{p}_{\boldsymbol{i}}\mathfrak{p}_{\boldsymbol{j}} \geq \mathfrak{p}_{\boldsymbol{i}} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^n \mu_{\mathfrak{p}}(B(x, r)).$$

Since $r/\rho = 2^n$, this can be rewritten to obtain

$$\mu_{\mathfrak{p}}(B(x, r)) \leq 3^n \mu_{\mathfrak{p}}(B(\xi, \rho)) = (2^n)^{\frac{\log(3)}{\log(2)}} \mu_{\mathfrak{p}}(B(\xi, \rho)) = \left(\frac{r}{\rho}\right)^{\frac{\log(3)}{\log(2)}} \mu_{\mathfrak{p}}(B(\xi, \rho)),$$

where equality can be obtained for appropriate choices of $B(\xi, \rho)$. Hence

$$Q := \dim_{\mathrm{As}}((\mathbb{Z}_2, \mu_{\mathfrak{p}})) = \frac{\log(3)}{\log(2)}. \tag{4.2.1}$$

Example 4.20 shows that even singleton points in $\mathbb{Q}_p$ possess nontrivial complex dimensions. As noted in that section, the translation invariance of the Haar measure on $\mathbb{Q}_p$ ensures that every singleton set is identical with respect to its complex dimensions. In contrast, $\mu_{\mathfrak{p}}$ is not translation invariant, and singleton sets in $(\mathbb{Z}_2, \mu_{\mathfrak{p}})$ don't possess this kind of dimensional uniformity.

The singleton sets $\{0\}$ and $\{1\}$ provide extremal examples. Zero is the fixed point of φ_0, and is therefore the limit of any sequence of points obtained by iterating φ_0. That is, if x is any point in $\mathbb{Z}_2$,

then $0 = \lim_{n\to\infty} \varphi_0^n(x)$ for any $x \in \mathbb{Z}_2$. Hence the distance zeta function $\zeta_{\{0\}}$ is given by

$$\begin{aligned}
\zeta_{\{0\}}(s) &= \int_{\mathbb{Z}_2} d(x,0)^{s-Q}\, \mathrm{d}\mu_{\mathfrak{p}}(x) \\
&= \sum_{n=0}^{\infty} \int_{\varphi_0^n(\mathbb{Z}_2)\setminus\varphi_0^{n+1}(\mathbb{Z}_2)} d(x,a)^{s-Q}\, \mathrm{d}\mu_{\mathfrak{p}}(x) \\
&= \sum_{n=0}^{\infty} \int_{\varphi_0^n(\mathbb{Z}_2)\setminus\varphi_0^{n+1}(\mathbb{Z}_2)} 2^{n(Q-s)}\, \mathrm{d}\mu_{\mathfrak{p}}(x) \\
&= \sum_{n=0}^{\infty} 2^{n(Q-s)} \left[\mu_{\mathfrak{p}}(\varphi_0^n(\mathbb{Z}_2)) - \mu_{\mathfrak{p}}(\varphi_0^{n+1}(\mathbb{Z}_2))\right] \\
&= \sum_{n=0}^{\infty} 2^{n(Q-s)} \left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^{n+1}\right] \\
&= \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{2^{Q-s}}{3}\right)^n \\
&= \frac{2}{3 - 2^{Q-s}}.
\end{aligned}$$

This is mentire with poles occurring whenever $2^{Q-s} = 3$, where $Q = \frac{\log(2)}{\log(3)}$ as in (4.2.1). Solving for s,

$$2^s = \frac{2^Q}{3} = \frac{2^{\frac{\log(3)}{\log(2)}}}{3} = 1 \implies s = \mathbb{i}\frac{2\pi k}{\log(2)},$$

where k is any integer. Therefore

$$\mathscr{P}(\zeta_{\{0\}}) = \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(2)}.$$

Similarly, 1 is the fixed point of φ_1, and $1 = \lim_{n\to\infty} \varphi_1^n(x)$ for any $x \in \mathbb{Z}_2$. Hence, by an almost identical computation,

$$\begin{aligned}
\zeta_{\{1\}}(s) &= \sum_{n=0}^{\infty} \int_{\varphi_1^n(\mathbb{Z}_2)\setminus\varphi_1^{n+1}(\mathbb{Z}_2)} 2^{n(Q-s)}\, \mathrm{d}\mu_{\mathfrak{p}}(x) \\
&= \sum_{n=0}^{\infty} 2^{n(Q-s)} \left[\mu_{\mathfrak{p}}(\varphi_1^n(\mathbb{Z}_2)) - \mu_{\mathfrak{p}}(\varphi_1^{n+1}(\mathbb{Z}_2))\right] \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2^{Q+1-s}}{3}\right)^n \\
&= \frac{1}{3 - 2^{Q+1-s}}.
\end{aligned}$$

This is mentire with poles occurring whenever $2^{Q+1-s} = 3$, where $Q = \frac{\log(2)}{\log(3)}$ as in (4.2.1). Again solving for s,

$$2^s = \frac{2^{Q+1}}{3} = \frac{2^{\frac{\log(3)}{\log(2)}} \cdot 2}{3} = 2 \implies s = 1 + \mathbb{i}\frac{2\pi k}{\log(2)},$$

where k is any integer. Therefore

$$\mathscr{P}(\zeta_{\{1\}}) = 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(2)}.$$

A possible interpretation of this result is that $\{0\}$ is a zero-dimensional fractal subset of $(\mathbb{Z}_2, \mu_{\mathfrak{p}})$, while $\{1\}$ is a one-dimensional fractal subset. A more compelling interpretation is offered in Chapter 5, in which the preceding computations can be recast in terms of the local distance zeta functions at zero and one. ◁

4.3 Examples in Vector Spaces over $\mathbb{Q}_p$

In this section, several examples of sets that occur as the attractors of self-similar iterated function systems on $\mathbb{Q}_p^Q$ are given, where $Q \geq 1$ is a natural number. A slightly modified definition is used: a self-similar contraction on $\mathbb{Q}_p^Q$ is a map of the form

$$\varphi(\boldsymbol{x}) = p^k \boldsymbol{x} + \boldsymbol{b},$$

where $k \in \mathbb{N}$ and $\boldsymbol{b} \in \mathbb{Q}_p^Q$. This is a somewhat more restrictive definition of a self-similar contraction than that introduced previously.

In particular, Hutchinson [Hut81] observes that if $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is a contracting similitude (as per Definition 2.11), then

$$\psi(\boldsymbol{x}) = cU\boldsymbol{x} + \boldsymbol{b},$$

where U is a unitary (orthogonal) matrix, c is the contraction ratio of ψ, and $\boldsymbol{b}$ is a translation. Thus a self-similar contraction mapping consists of a scaling (the contraction), a translation, and an unitary transformation (a rotation and/or reflection). In the above definition of a self-similar contraction

mapping on $\mathbb{Q}_p^Q$, any analog of a unitary transformation has been omitted, and the contraction ratio is assumed to be of the form p^{-k}.

The second restriction is no restriction at all: if φ is a contracting similitude on $\mathbb{Q}_p^Q$, then there is some $k \in \mathbb{N}$ such that

$$d(\varphi(x), \varphi(y)) = |\varphi(x) - \varphi(y)|_p = p^{-k}$$

for any $x, y \in \mathbb{Q}_p^Q$. The metric structure of $\mathbb{Q}_p^Q$ ensures that every contraction ratio must be a power of the prime p. The omission of unitary transformations greatly simplifies computation, and, due to the non-archimedean nature of the metric on $\mathbb{Q}_p^Q$, represents no great loss of generality.

Self-similar sets with contraction ratio p

Proposition 4.23. *Let $\mathscr{I} \subsetneq \{0, 1, \ldots, p-1\}^Q$, and for each $\boldsymbol{i} \in \mathscr{I}$ define the map*

$$\varphi_{\boldsymbol{i}} : \mathbb{Q}_p^Q \to \mathbb{Q}_p^Q \qquad \text{where} \qquad \varphi_{\boldsymbol{i}}(\boldsymbol{x}) := p\boldsymbol{x} + \boldsymbol{i}.$$

Note that each such map is a contracting similitude with ratio p^{-1}. The collection of maps $\{\varphi_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathscr{I}}$ defines a self-similar iterated function system with attractor $\mathscr{A}$. Then the complex dimensions of $\mathscr{A}$ are given by

$$\mathscr{P}(\zeta_{\mathscr{A}}) = \left\{ \frac{\log |\mathscr{I}|}{\log(p)} + \mathbb{i} \frac{2\pi\mathbb{Z}}{\log(p)} \right\}$$

This result is independent of the choice of metric (that is, for any $\alpha \in [1, \infty]$, the result holds).

Proof. It will be convenient to work within the framework of relative fractal drums. Let

$$\Omega := B_{\leq}(0, 1) \setminus \mathscr{A} = \mathbb{Z}_p^Q \setminus \mathscr{A},$$

and for each $n \in \mathbb{N}$, define

$$\Omega_0 := \mathbb{Z}_p^Q \setminus \Phi(\mathbb{Z}_p^Q), \qquad \text{and} \qquad \Omega_n := \Phi(\Omega_{n-1}) = \Phi^n(\Omega_0).$$

If $m \neq n$ then $\Omega_m \cap \Omega_n = \varnothing$, and $\Omega = \bigcup \Omega_n$. Then, by definition of the union of relative fractal drums,

$$(\mathscr{A}, \Omega) = \bigcup_{n=0}^{\infty} (\mathscr{A}, \Omega_n)\,.$$

This union satisfies the hypotheses of Theorem 3.34 (with $A = A_n = \mathscr{A}$ for all n), hence

$$\zeta_{\mathscr{A}}(s) = \zeta_{\mathscr{A},\Omega}(s) = \sum_{n=0}^{\infty} \zeta_{\mathscr{A},\Omega_n}(s). \tag{4.3.1}$$

Next, by definition of Ω_n

$$\Omega_n = \Phi(\Omega_{n-1}) = \bigcup_{\boldsymbol{i} \in \mathscr{I}} (p\Omega_{n-1} + \boldsymbol{i})\,.$$

But $p\Omega_{n-1} + \boldsymbol{i}$ is isometric to $p\Omega_{n-1}$, hence Ω_n is the (disjoint) union of $|\mathscr{I}|$ copies of $p\Omega_{n-1}$ (where $|\mathscr{I}|$ denotes the cardinality of $\mathscr{I}$). Similarly, as $\mathscr{A}$ is the attractor of $\{\varphi_{\boldsymbol{i}}\}$, it follows that

$$\mathscr{A} = \Phi(\mathscr{A}) = \bigcup_{\boldsymbol{i} \in \mathscr{I}} (p\mathscr{A} + \boldsymbol{i})\,,$$

which is the union of $|\mathscr{I}|$ copies of $p\mathscr{A}$. It then follows from Theorems 3.34 and 3.32 that for each $n \in \mathbb{N}$,

$$\zeta_{\mathscr{A},\Omega_n}(s) = \sum_{\boldsymbol{i} \in \mathscr{I}} \zeta_{p\mathscr{A},p\Omega_{n-1}}(s) = \sum_{\boldsymbol{i} \in \mathscr{I}} p^{-s} \zeta_{\mathscr{A},\Omega_{n-1}}(s) = |\mathscr{I}| p^{-s} \zeta_{\mathscr{A},\Omega_{n-1}}(s).$$

By induction, it follows that

$$\zeta_{\mathscr{A},\Omega_n}(s) = (|\mathscr{I}| p^{-s})^n \, \zeta_{\mathscr{A},\Omega_0}(s).$$

Finally, note that if $\boldsymbol{x} \in \Omega_0$, then $d(\boldsymbol{x}, \mathscr{A}) = 1$. It then follows that

$$\begin{aligned}\zeta_{\mathscr{A},\Omega_0}(s) &= \int_{\Omega_0} d(\boldsymbol{x}, \mathscr{A})^{s-Q} \, \mathrm{d}\mu(\boldsymbol{x}) = \int_{\Omega_0} \mathrm{d}\mu(\boldsymbol{x}) = \mu(\Omega_0) = \mu(\mathbb{Z}_p^Q \setminus \Phi(\mathbb{Z}_p^Q) \\ &= \mu(\mathbb{Z}_p^Q) - \mu\left(\bigcup_{\boldsymbol{i} \in \mathscr{I}} \varphi_{\boldsymbol{i}}\left(\mathbb{Z}_p^Q\right)\right) = 1 - \sum_{\boldsymbol{i} \in \mathscr{I}} \mu\left(p\mathbb{Z}_p^Q\right) = 1 - |\mathscr{I}| p^{-Q}. \end{aligned} \tag{4.3.2}$$

Therefore (4.3.1) becomes

$$\zeta_{\mathscr{A}}(s) = \sum_{n=0}^{\infty} \zeta_{\mathscr{A},\Omega_n}(s) = \sum_{n=0}^{\infty} \left(|\mathscr{I}|p^{-s}\right) \zeta_{\mathscr{A},\Omega_0}(s) = \left(1 - \frac{|\mathscr{I}|}{p^Q}\right) \sum_{n=0}^{\infty} \left(\frac{|\mathscr{I}|}{p^s}\right)^n.$$

Observe that this series converges on the open right half-plane $\{\Re(s) > \log_p |\mathscr{I}|\}$. On this half-plane,

$$\zeta_{\mathscr{A}}(s) = \left(1 - \frac{|\mathscr{I}|}{p^Q}\right) \sum_{n=0}^{\infty} \left(\frac{|\mathscr{I}|}{p^s}\right)^n = \left(1 - \frac{|\mathscr{I}|}{p^Q}\right) \frac{p^s}{p^s - |\mathscr{I}|} \tag{4.3.3}$$

which is a mentire function that extends $\zeta_{\mathscr{A}}$ to all of $\mathbb{C}$. It therefore follows that the complex dimensions of $\mathscr{A}$ are given by

$$\mathscr{P}(\zeta_{\mathscr{A}}) = \left\{ \frac{\log |\mathscr{I}|}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)} \right\}.$$

Note that in the p-adic setting, the complex dimensions of the attractor of a self-similar iterated function system on $\mathbb{Q}_p^Q$ depend only on p and the number of maps in the system. In particular, the complex dimensions do not depend on the ambient dimension Q (except in the sense that p^Q gives an upper bound on $|\mathscr{I}|$).

It is also worth noting that this result does not depend on the choice of metric. If, instead, $\{\varphi_{\boldsymbol{i}}\}$ is regarded as an iterated function system on $(\mathbb{Q}_p^Q, d^\alpha)$ for some $\alpha \in [1, \infty)$, then $d(\boldsymbol{x}, \mathscr{A}) = Q^{1/\alpha}$ for any $\boldsymbol{x} \in \Omega_0$. Hence (4.3.2) becomes

$$\zeta_{\mathscr{A},\Omega_0}(s) = \int_{\Omega_0} d(\boldsymbol{x}, \mathscr{A})^{s-Q} \, \mathrm{d}\mu(\boldsymbol{x}) = Q^{1/\alpha} \mu(\Omega_0) = Q^{1/\alpha} \left(1 - \frac{|\mathscr{I}|}{p^Q}\right).$$

The distance zeta function is then given by

$$\zeta_{\mathscr{A}}(s) = Q^{1/\alpha} \left(1 - \frac{|\mathscr{I}|}{p^Q}\right) \frac{p^s}{p^s - |\mathscr{I}|},$$

which is mentire and has the same pole set as (4.3.3). □

The following two examples are applications of Example 4.23 in cases where the complex dimensions have previously been computed using other techniques.

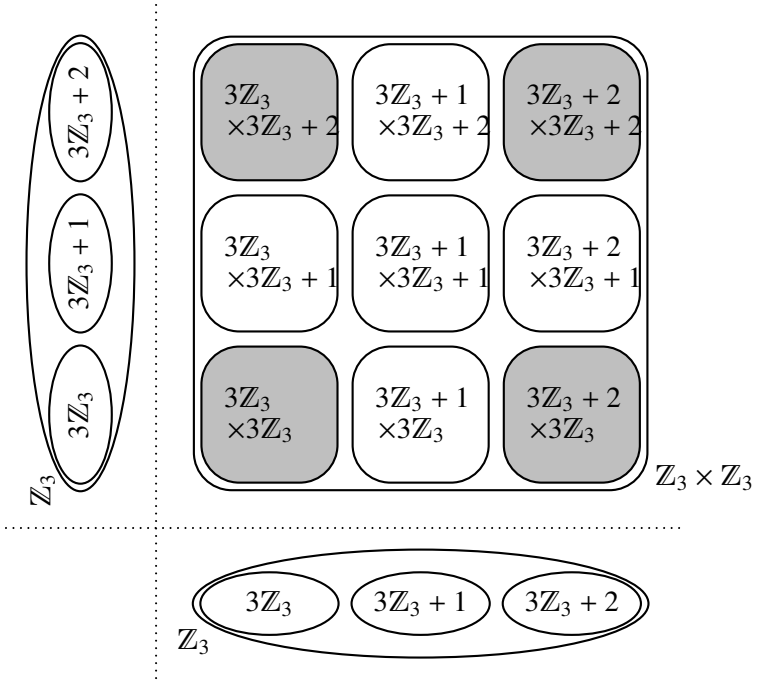


Figure 4.2: A schematic representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The iterated function system which gives rise to the 3-adic Cantor dust consists of the four maps which take $\mathbb{Z}_3$ to each of the four shaded rectangles in the product space. For example, $\varphi_{(0,2)}(\mathbb{Z}_3) = (3\mathbb{Z}_3) \times (3\mathbb{Z}_3 + 2)$.

Example 4.24. Let $\mathscr{C}_3$ denote the 3-adic Cantor set, which is the attractor of an iterated function system on $\mathbb{Q}_3$ consisting of two maps, each with a scaling ratio of 3. Specifically, the maps are given by

$$\varphi_0(x) = 3x, \qquad \text{and} \qquad \varphi_2(x) = 3x + 2,$$

where $x \in \mathbb{Q}_3$. By Proposition 4.23,

$$\mathscr{P}(\zeta_{\mathscr{C}_3}) = \frac{\log(2)}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}.$$

◁

Example 4.25. The Cartesian product of two copies of the 3-adic Cantor set is the 3-adic Cantor dust, denoted by $\mathscr{C}_3^2$. This set may be realized as the attractor of an iterated function system consisting of

four maps, each with contration ratio $1/3$. For each

$$j \in \{(0,0),(0,2),(2,0),(2,2)\},$$

define the similitude

$$\varphi_j : \mathbb{Q}_3^2 \to \mathbb{Q}_3^2$$

by

$$\varphi_j(x) = 3x + j.$$

See Figure 4.2 for a representation of the action of these maps on the dressed unit ball $\mathbb{Z}_3$. It follows from Proposition 4.23 that

$$\mathscr{P}(\zeta_{\mathscr{C}_3^2}) = \frac{\log(4)}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)},$$

which is the expected result. ◁

General self-similar sets

Let $\{\varphi_i\}_{i\in\mathscr{I}}$ be a self-similar IFS on $\mathbb{Q}_p^Q$ indexed by some finite set $\mathscr{I}$. As indicated above, it may be assumed without loss of generality that each map is of the form

$$\varphi_i(\boldsymbol{x}) = p^{k_i}\boldsymbol{x} + \boldsymbol{b}_i,$$

where $k_i \in \mathbb{N}$ and $\boldsymbol{b}_i \in \mathbb{Z}_p^Q$ for all $i \in \mathscr{I}$. Assume further that

$$\varphi_i(\mathbb{Z}_p^Q) \cap \varphi_j(\mathbb{Z}_p^Q) \qquad \text{for all } i,j \in \mathscr{I} \text{ with } i \neq j. \tag{4.3.4}$$

That is, assume that $\{\varphi_i\}_{i\in\mathscr{I}}$ satisfies the open set condition with open set $\mathbb{Z}_p^Q$. Let

$$K := \max\left\{k \in \mathbb{N} \;\middle|\; \varphi_i(\boldsymbol{x}) = p^k\boldsymbol{x} + \boldsymbol{b}_i \text{ for some } i \in \mathscr{I}\right\}.$$

As $\mathscr{I}$ is a finite index set, K is well defined. For each $k = 1, \ldots, K$, let

$$c_k := \left|\left\{i \in \mathscr{I} \;\middle|\; \varphi_i(\boldsymbol{x}) = p^k \boldsymbol{x} + \boldsymbol{b}_i\right\}\right|.$$

That is, c_k denotes the number of maps in the iterated function system $\{\varphi_i\}_{i\in\mathscr{I}}$ with the contraction ratio p^{-k}. For each $n \in \mathbb{Z}$, let

$$C_n := \left|\{\boldsymbol{i} \in \mathscr{I}^* \mid \varphi_{\boldsymbol{i}}(\boldsymbol{x}) = p^n \boldsymbol{x} + \boldsymbol{b}_{\boldsymbol{i}}\}\right|.$$

That is, C_n is counts the number of ways that maps from $\{\varphi_i\}_{i\in\mathscr{I}}$ can be composed in order to obtain a map with contraction ratio p^{-n}. Observe that the values of C_n are given by the linear, homogeneous, K-th order recurrence relation

$$C_n = \sum_{k=1}^{K} c_k C_{n-k},$$

with the initial values

$$C_n = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0 \text{ (this counts the empty composition), and} \\ c_1 & \text{if } n = 1. \end{cases}$$

While explicit solutions to such recurrence relations can be determined in principle, such solutions are unenlightening in the current context.

Let $\mathscr{A}$ denote the attractor of this IFS, and define

$$\Omega := \mathbb{Z}_p^Q \setminus \mathscr{A} \qquad \text{and} \qquad \Omega_\iota := \mathbb{Z}_p^Q \setminus \Phi(\mathbb{Z}_p^Q).$$

For each $\boldsymbol{i} \in \mathscr{I}^*$, let

$$\Omega_{\boldsymbol{i}} := \varphi_{\boldsymbol{i}}(\Omega_\iota) = \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_{|\boldsymbol{i}|}}(\Omega_\iota).$$

Note that

$$\Omega = \bigcup_{\boldsymbol{i} \in \mathscr{I}^*} \varphi_{\boldsymbol{i}}(\Omega_\iota).$$

where the condition (4.3.4) ensures that this union is disjoint. Since $\mathbb{Z}_p^Q$ is a δ-neighborhood of $\mathscr{A}$, it follows from Theorem 3.34 that

$$\zeta_{\mathscr{A}}(s) = \zeta_{\mathscr{A},\Omega}(s) = \sum_{\boldsymbol{i} \in \mathscr{I}^*} \zeta_{\varphi_{\boldsymbol{i}}(\mathscr{A}),\varphi_{\boldsymbol{i}}(\Omega_\iota)}(s). \tag{4.3.5}$$

Since the composition of similarities is also a similarity, it follows from Theorem 3.32 that

$$\zeta_{\varphi_{\boldsymbol{i}}(\mathscr{A}),\varphi_{\boldsymbol{i}}(\Omega_\iota)}(s) = \left(p^{-\sum_{i=1}^{|\boldsymbol{i}|} k_i}\right)^s \zeta_{\mathscr{A},\Omega_\iota}(s),$$

hence (4.3.5) becomes

$$\begin{aligned}
\sum_{\boldsymbol{i} \in \mathscr{I}^*} \zeta_{\varphi_{\boldsymbol{i}}(\mathscr{A}),\varphi_{\boldsymbol{i}}(\Omega_\iota)}(s) &= \sum_{\boldsymbol{i} \in \mathscr{I}^*} \left(p^{-\sum_{i=1}^{|\boldsymbol{i}|} k_i}\right)^s \zeta_{\mathscr{A},\Omega_\iota}(s) \\
&= \zeta_{\mathscr{A},\Omega_\iota}(s) \sum_{\boldsymbol{i} \in \mathscr{I}^*} p^{-s\sum_{i=1}^{|\boldsymbol{i}|} k_i} \\
&= \zeta_{\mathscr{A},\Omega_\iota}(s) \sum_{n=0}^{\infty} C_n p^{-ns}
\end{aligned}$$

In summary, if $\{\varphi_i\}_{i \in \mathscr{I}}$ is a self-similar iterated function system on $\mathbb{Q}_p^Q$ with attractor $\mathscr{A}$, then the associated distance zeta function is given by

$$\zeta_{\mathscr{A}}(s) = \zeta_{\mathscr{A},\Omega_\iota}(s) \sum_{n=0}^{\infty} C_n p^{-ns}, \tag{4.3.6}$$

where $\zeta_{\mathscr{A},\Omega_\iota}(s)$ and $\{C_n \mid n \in \mathbb{N}\}$ can be given explicitly in particular examples. Moreover, due to the discrete nature of the family of metrics d^α on $\mathbb{Q}_p^Q$, the function $\zeta_{\mathscr{A},\Omega_\iota}(s)$ will always be entire. Hence this term contributes no singularities to the distance zeta function and, at worst, may lead to cancelation of singularities. This implies that the complex dimensions are $\mathscr{A}$ are a subset of the poles of the mentire function that extends the series in (4.3.6).

Example 4.26. On $\mathbb{Q}_p$, let

$$\varphi_1(x) := px, \qquad \text{and} \qquad \varphi_2(x) := p^2x + 1.$$

Retaining the notation developed above,

$$\begin{aligned}
\zeta_{\mathscr{A},\Omega_\iota}(s) &= \int_{\Omega_\iota} d(x,\mathscr{A})^{s-1}\,\mathrm{d}\mu(x) \\
&= \sum_{j=2}^{p-1}\int_{p\mathbb{Z}_p+j} d(x,\mathscr{A})^{s-1}\,\mathrm{d}\mu(x) + \sum_{j=1}^{p-1}\int_{p^2\mathbb{Z}_p+pj+1} d(x,\mathscr{A})^{s-1}\,\mathrm{d}\mu(x) \\
&+ \sum_{j=2}^{p-1}\int_{p\mathbb{Z}_p+j} \mathrm{d}\mu(x) + \sum_{j=1}^{p-1}\int_{p^2\mathbb{Z}_p+pj+1} (p^{-1})^{s-1}\,\mathrm{d}\mu(x) \\
&= \sum_{j=2}^{p-1}\mu(p\mathbb{Z}_p+j) + \sum_{j=1}^{p-1} p^{1-s}\mu(p^2\mathbb{Z}_p+pj+1) \\
&= \frac{p-2}{p} + \frac{p-1}{p^2}p^{1-s}.
\end{aligned}$$

To determine the values of C_n, note that $c_1 = c_2 = 1$, and so

$$C_n = C_{n-1} + C_{n-2}, \qquad \text{with initial values } C_0 = 1 \text{ and } C_1 = 1.$$

This is the recurrence relation defining the Fibonacci sequence, which has the well-known closed form

$$C_n = F_{n+1} = \frac{1}{\sqrt{5}}\left(\phi^{n+1} + \psi^{n+1}\right),$$

where ϕ and ψ are the golden ratio and its conjugate, given by

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \psi = \frac{1-\sqrt{5}}{2}.$$

Substituting this into (4.3.6) gives

$$\zeta_{\mathscr{A}}(s) = \frac{1}{\sqrt{5}}\left(\frac{p-2}{p} + \frac{p-1}{p^2}p^{1-s}\right)\sum_{n=0}^{\infty}\left(\phi^{n+1} - \psi^{n+1}\right)p^{-ns}. \tag{4.3.7}$$

The series converges absolutely for all s with sufficiently large real portion. For such s,

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\phi^{n+1}-\psi^{n+1}\right)p^{-ns} &= \phi\sum_{n=0}^{\infty}\left(\frac{\phi}{p^s}\right)^n - \psi\sum_{n=0}^{\infty}\left(\frac{\psi}{p^s}\right)^n \\ &= \frac{\phi p^s}{p^s-\phi} - \frac{\psi p^s}{p^s-\psi} \\ &= \frac{\phi p^s(p^s-\psi)-\psi p^s(p^s-\phi)}{(p^s-\phi)(p^s-\psi)} \\ &= \frac{(\phi-\psi)p^{2s}}{(p^s-\phi)(p^s-\psi)} \\ &= \frac{\sqrt{5}p^{2s}}{(p^s-\phi)(p^s-\psi)},\end{aligned}$$

which is mentire. Therefore, substituting this into (4.3.7), an explicit mentire extension of the distance zeta function is given by

$$\zeta_{\mathscr{A}}(s) = p\frac{(p-2)p^{2s}+(p-1)p^s}{(p^s-\phi)(p^s-\psi)}.$$

The numerator and denominator are never simultaneously zero, and so the poles of $\zeta_{\mathscr{A}}$ are exactly the zeros of the denominator. In particular,

$$\begin{aligned}\mathscr{P}(\zeta_{\mathscr{A}}) &= \frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}, \frac{\log(\psi)}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)} \\ &= \frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}, -\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{(2\pi+1)\mathbb{Z}}{\log(p)}\end{aligned}$$

In the special case $p = 2$, the corresponding attractor $\mathscr{A}$ is the 2-adic Fibonacci string described in [LvF13, Ex. 13.102]. ◁

A self-affine example

In general, computations involving self-affine (rather than self-similar) sets are much more difficult. In a general setting, contractions which scale by log-incommensurable ratios in different directions provide a significant obstruction. In vector spaces over $\mathbb{Q}_p$ (where p is fixed), this obstruction does not exist. The following example describes a self-affine set in $\mathbb{Q}_3^2$.

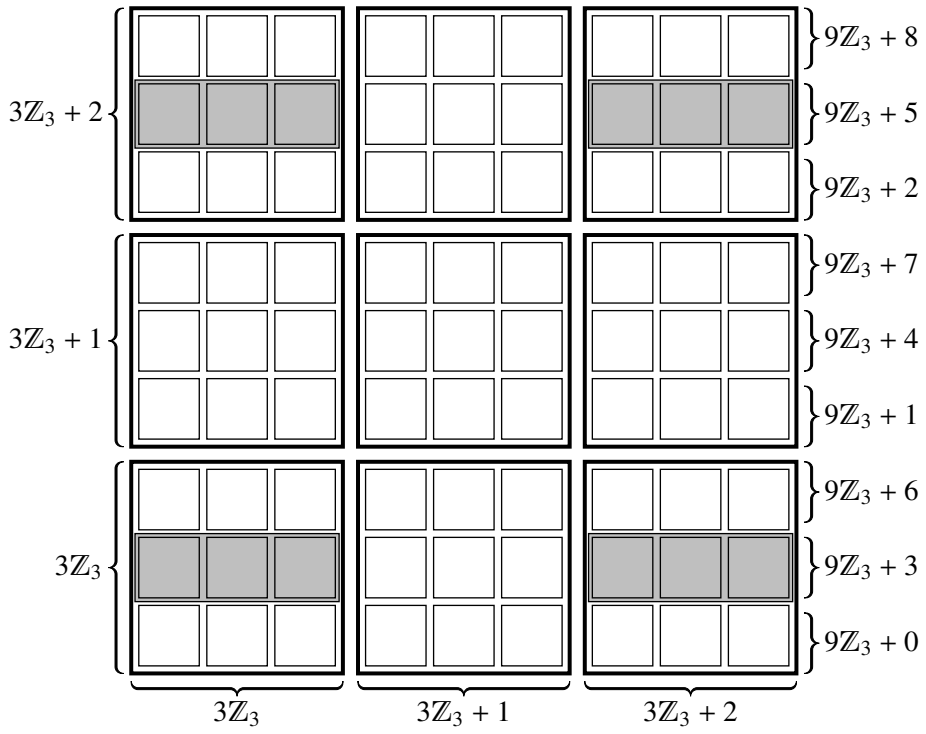


Figure 4.3: The unit "square" in $\mathbb{Q}_3^2$ is the set $\mathbb{Z}_3^2$. This square consists of 9 squares of the form $(3\mathbb{Z}_3 + i) \times (3\mathbb{Z}_3 + j)$, where $i, j = 0, 1, 2$. Each of these, in turn, consists of 9 smaller squares, and so on. The image of $\mathbb{Z}_3^2$ under the IFS is shown in grey.

Example 4.27. Let $\{\varphi_i\}_{i=1}^4$ be the IFS on $\mathbb{Q}_3^2$ consisting of the maps

$$\varphi_1(\boldsymbol{x}) = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad \varphi_2(\boldsymbol{x}) = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$\varphi_3(\boldsymbol{x}) = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \qquad \varphi_4(\boldsymbol{x}) = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

The action of this IFS on $\mathbb{Z}_3^2$ is diagramed in Figure 4.3. Let $\mathscr{A}$ denote the attractor of this system, let $\Omega = \mathbb{Z}_3^2 \setminus \mathscr{A}$, and let

$$A_0 := \mathbb{Z}_3^2 \qquad \text{and} \qquad A_n := \Phi^n(A_0)$$

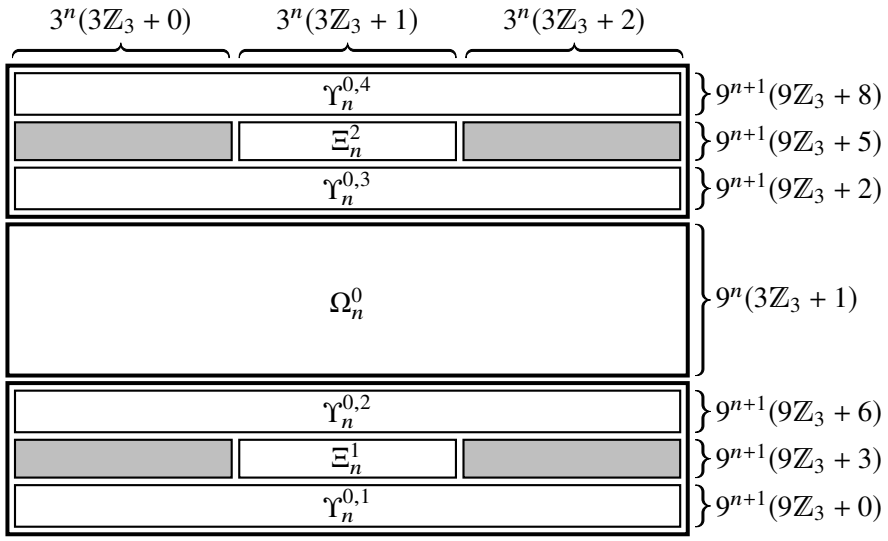


Figure 4.4: A decomposition of $\varphi_1^n(\mathbb{Z}_3^2) = 3^n\mathbb{Z}_3 \times 9^n\mathbb{Z}_3$. Each of the 4^n congruent rectangles that compose $A_n \setminus A_{n+1}$ can be similarly decomposed into sets Ω_n^i, $\Upsilon_n^{i,j}$, and $\Xi_n^{i,j}$. To simplify notation, let $\Omega_n := \Omega_n^0$, $\Upsilon_n := \Upsilon_n^{0,1}$, and $\Xi_n := \Xi_n^{0,1}$.

denote the prefractal approximates. Observe that A_n consists of 4^n translated copies of the rectangle $3^n\mathbb{Z}_3 \times 9^n\mathbb{Z}_3$, and that each of these rectangles can be further decomposed into the rectangles Ω_n, four sets of the form $\Upsilon_n^{i,j}$, and two sets of the form $\Xi_n^{i,j}$, as shown in Figure 4.4. As all of these sets are disjoint,

$$(\mathscr{A},\Omega) = \bigcup_{n=0}^{\infty}(\mathscr{A},A_n\setminus A_{n+1}) = \bigcup_{n=0}^{\infty}\bigcup_{i=0}^{4^n-1}(\mathscr{A},\Omega_n^i)\cup\bigcup_{n=0}^{\infty}\bigcup_{i=0}^{4^n-1}\bigcup_{j=1}^{4}(\mathscr{A},\Upsilon_n^{i,j})\cup\bigcup_{n=0}^{\infty}\bigcup_{i=0}^{4^n-1}\bigcup_{j=1}^{2}(\mathscr{A},\Xi_n^{i,j}).$$

Hence

$$\begin{aligned}\zeta_{\mathscr{A}}(s) &= \sum_{n=0}^{\infty}\sum_{i=0}^{4^n-1}\left[\zeta_{\mathscr{A},\Omega_n^i}(s) + \sum_{j=1}^{4}\zeta_{\mathscr{A},\Upsilon_n^{i,j}}(s) + \sum_{j=1}^{2}\zeta_{\mathscr{A},\Xi_n^{i,j}}(s)\right]\\ &= \sum_{n=0}^{\infty}4^n\left[\zeta_{\mathscr{A},\Omega_n}(s) + 4\zeta_{\mathscr{A},\Upsilon_n}(s) + 2\zeta_{\mathscr{A},\Xi_n}(s)\right],\\ &= \sum_{n=0}^{\infty}4^n\zeta_{\mathscr{A},\Omega_n}(s) + 4\sum_{n=0}^{\infty}4^n\zeta_{\mathscr{A},\Upsilon_n}(s) + 2\sum_{n=0}^{\infty}4^n\zeta_{\mathscr{A},\Xi_n}(s) \qquad (4.3.8)\end{aligned}$$

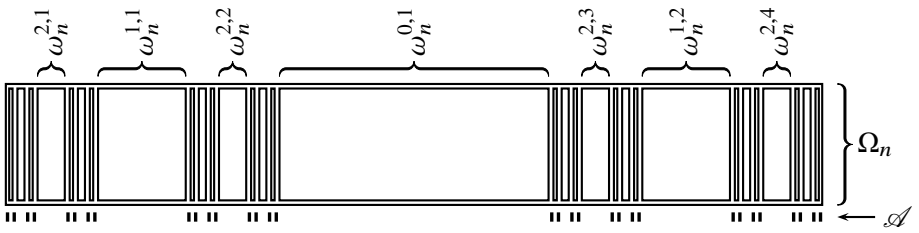


Figure 4.5: The RFD Ω_n can be decomposed into sub-RFDs as shown here. Note that $\bigcup_{m,k} \omega_n^{m,k}$ is the complement of $9^n(3\mathbb{Z}_3 + 1) \times \mathscr{C}$, where $\mathscr{C}$ is a ternary Cantor set in $\mathbb{Z}_3$.

---

where Ω_n may be taken to be any of the 4^n congruent sets of the form Ω_n^i contained in $\mathscr{A}_n \setminus A_{n+1}$ (Υ_n and Ξ_n can be defined similarly).

If $\boldsymbol{x} \in \Xi_n$, then the "horizontal" distance from $\boldsymbol{x}$ to the attractor is exactly 3^{-n}, while the "vertical" distance to the attractor is at most 9^{-n}. Thus $d(\boldsymbol{x}, \Xi_n) = 3^{-n}$ for any such $\boldsymbol{x}$. As $\mu(\Xi_n) = 27^{-n-1}$, it follows that

$$\begin{aligned}\zeta_{\mathscr{A},\Xi_n}(s) = \int_{\Xi_n} d(\boldsymbol{x}, \mathscr{A})^{s-2}\, \mathrm{d}\mu(x) &= \int_{\Xi_n} \left(\frac{1}{3}\right)^{n(s-2)} \mathrm{d}\mu(x) \\ &= \left(\frac{1}{3}\right)^{n(s-2)} \mu(\Xi_n) = \left(\frac{1}{3}\right)^{n(s-2)} \left(\frac{1}{27}\right)^{n+1} = \frac{1}{3}\left(\frac{1}{3^{s+1}}\right)^n.\end{aligned}$$

Thus

$$2\sum_{n=0}^{\infty} 4^n \zeta_{\mathscr{A},\Xi_n}(s) = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{4}{3^{s+1}}\right)^n = \frac{2\cdot 3^s}{3\cdot 3^s - 4}. \tag{4.3.9}$$

Obtaining an explicit formula for $\zeta_{\mathscr{A},\Omega_n}(s)$ is somewhat more complicated. Further sub-divide Ω_n as shown in Figure 4.5. By construction of the $\omega_n^{m,k}$,

$$\Omega_n = (\text{a set of measure zero}) \cup \bigcup_{m=0}^{\infty}\bigcup_{k=1}^{2^m} \omega_n^{m,k}, \tag{4.3.10}$$

where the unions are disjoint. Note that if $\boldsymbol{x} \in \omega_n^{m,k}$, then the "vertical" distance to the attractor is 9^{-n}, and the "horizontal" distance is $3^{-(m+n)}$. Hence for any $\boldsymbol{x} \in \omega_n^{m,k}$,

$$d(\boldsymbol{x}, \mathscr{A}) = \max\left\{\left(\frac{1}{9}\right)^n, \left(\frac{1}{3}\right)^{m+n}\right\} = \begin{cases} \left(\frac{1}{3}\right)^{m+n} & \text{if } m < n, \text{ and} \\ \left(\frac{1}{3}\right)^{2n} & \text{if } m \geq n. \end{cases}$$

In particular, the distance does not depend on k. Next, note that

$$\mu(\omega_n^{m,k}) = \left(\frac{1}{3}\right)^m \mu(\Omega_n) = \left(\frac{1}{3}\right)^{m+3n+1}$$

Again, note that this does not depend on k. It then follows that

$$\begin{aligned} \zeta_{\mathscr{A},\Omega_n}(s) &= \sum_{m=0}^{\infty} \sum_{k=1}^{2^m} \zeta_{\mathscr{A},\omega_n^{m,k}}(s) \\ &= \sum_{m=0}^{\infty} 2^m \zeta_{\mathscr{A},\omega_n^{m,1}}(s) \\ &= \frac{3^{s-1}}{3^s - 6}\left(\frac{1}{3^{s+1}}\right)^n - \frac{3^{s-1}}{3^s - 6}\left(\frac{2}{3^{2s}}\right)^n + \left(\frac{2}{3^{2s}}\right)^n. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} 4^n \zeta_{\mathscr{A},\Omega_n}(s) &= \frac{3^{s-1}}{3^s - 6} \sum_{n=0}^{\infty}\left(\frac{4}{3^{s+1}}\right)^n - \frac{3^{s-1}}{3^s - 6} \sum_{n=0}^{\infty}\left(\frac{8}{3^{2s}}\right)^n + \sum_{n=0}^{\infty}\left(\frac{8}{3^{2s}}\right)^n \\ &= \frac{9 \cdot 3^{3s} - 8 \cdot 3^{2s}}{3(3 \cdot 3^s - 4)(3^{2s} - 8)}. \end{aligned} \tag{4.3.11}$$

Finally, note that Υ_n can be decomposed in a manner similar to Ω_n (replace Ω_n with Υ_n and $\omega_n^{m,k}$ with $\upsilon_n^{m,k}$ in Figure 4.5 to get an intuition for the decomposition). If $\boldsymbol{x} \in \upsilon_n^{m,k}$, then the "vertical" distance to the attractor is $3 \cdot 9^{-n}$, and the "horizontal" distance to the attractor is $3^{-(m+n)}$. Hence if

$\boldsymbol{x} \in \upsilon_n^{m,k}$, then

$$d(\boldsymbol{x}, \mathscr{A}) = \max\left\{\frac{1}{3}\left(\frac{1}{9}\right)^n, \left(\frac{1}{3}\right)^{m+n}\right\} = \begin{cases} \left(\frac{1}{3}\right)^{m+n} & \text{if } m \leq n, \text{ and} \\ \left(\frac{1}{3}\right)^{2n+1} & \text{if } m > n. \end{cases}$$

Also

$$\mu(\upsilon_n^{m,k}) = \left(\frac{1}{3}\right)^m \mu(\Upsilon_n) = \left(\frac{1}{3}\right)^{m+3n+2}$$

As was the case in the decomposition of Ω_n, none of these distances or volumes depends on k. It the follows that

$$\begin{aligned} \zeta_{\mathscr{A},\Upsilon_n}(s) &= \sum_{m=0}^{\infty}\sum_{k=1}^{2^m} \zeta_{\mathscr{A},\upsilon_n^{m,k}}(s) \\ &= \sum_{m=0}^{\infty} 2^m \zeta_{\mathscr{A},\upsilon_n^{m,1}}(s) \\ &= \frac{3^{s-2}}{3^s-6}\left(\frac{1}{3^{s+1}}\right)^n - \frac{2}{3(3^s-6)}\left(\frac{2}{3^{2s}}\right)^n + \frac{2}{3^s}\left(\frac{2}{3^{2s}}\right)^n. \end{aligned}$$

Hence

$$\begin{aligned} &4\sum_{n=0}^{\infty} 4^n \zeta_{\mathscr{A},\Upsilon_n}(s) \\ &\quad= \frac{4\cdot 3^{s-2}}{3^s-6}\sum_{n=0}^{\infty}\left(\frac{4}{3^{s+1}}\right)^n - \frac{8}{3(3^s-6)}\sum_{n=0}^{\infty}\left(\frac{8}{3^{2s}}\right)^n + \frac{8}{3^s}\sum_{n=0}^{\infty}\left(\frac{8}{3^{2s}}\right)^n \\ &\quad= \frac{4\cdot 3^{3s} + 72\cdot 3^{2s} - 96\cdot 3^s}{3(3\cdot 3^s-4)(3^{2s}-8)} \end{aligned} \tag{4.3.12}$$

Substituting the results of (4.3.9), (4.3.11), and (4.3.12) into the formula at (4.3.8), the distance zeta function is explicitly given by

$$\begin{aligned}\zeta_{\mathscr{A}}(s) &= \sum_{n=0}^{\infty} 4^n \zeta_{\mathscr{A},\Omega_n}(s) + 4\sum_{n=0}^{\infty} 4^n \zeta_{\mathscr{A},\Upsilon_n}(s) + 2\sum_{n=0}^{\infty} 4^n \zeta_{\mathscr{A},\Xi_n}(s) \\ &= \frac{9\cdot 3^{3s} - 8\cdot 3^{2s}}{3(3\cdot 3^s - 4)(3^{2s}-8)} + \frac{4\cdot 3^{3s} + 72\cdot 3^{2s} - 96\cdot 3^s}{3(3\cdot 3^s - 4)(3^{2s}-8)} + \frac{2\cdot 3^s}{3\cdot 3^s - 4} \\ &= \frac{3^s(5\cdot 3^{2s} + 16\cdot 3^s - 32)}{(3\cdot 3^s - 4)(3^{2s}-8)}.\end{aligned}$$

This function may be extended to a mentire function with poles

$$\mathscr{P}(\zeta_{\mathscr{A}}) = \frac{\log(4)}{\log(3)} + 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}, \frac{3\log(2)}{2\log(3)} + \mathbb{i}\frac{\pi\mathbb{Z}}{\log(3)}.$$

Observe that the principle complex dimension $\frac{3}{2}\log_3(2)$ is exactly what might be expected for the $\mathbb{R}^2$ analog of $\mathscr{A}$ (that is, the product of the $\frac{1}{3}$- and $\frac{1}{9}$-Cantor sets in $\mathbb{R}^2$), as described in [McM84]. ◁

4.4 Comparison with Past Results

A theory of self-similar p-adic fractal strings was developed by Lapidus and Lũ' [LL08, LL09]. This theory closely parallels the one-dimensional theory of fractal harps in $\mathbb{R}$ described by Lapidus and van Frankenjuijsen [LvF13], and recalled above in Chapter 1.

Let $\mathcal{L}_p \subseteq \mathbb{Q}_p$ be open and bounded. The set $\mathcal{L}_p$ may be written as

$$\bigcup_{j\in\mathbb{N}} a_j + p^{n_j}\mathbb{Z}_p = \bigcup_{j\in\mathbb{N}} B(a_j, p^{-n_j})$$

where for any point $a_j \in \mathcal{L}_p$, the ball $B(a_j, p^{-n_j})$ is the largest dressed ball containing a_j which is contained in $\mathcal{L}_p$. This decomposition is not unique, but the boundedness of $\mathcal{L}_p$ together with the fact that "every point is the center" (see Theorem 4.18(c)) ensures that there is a unique minimal decomposition into non-overlapping balls.

The *p-adic geometric zeta function* associated to $\mathcal{L}_p$ is

$$\zeta_{\mathcal{L}_p}(s) := \sum_{j\in\mathbb{N}} \mu_p(B(a_j, p^{-n_j})^s = \sum_{j\in\mathbb{N}} p^{-n_j s}.$$

In the framework of the distance and tube zeta functions developed here, study a p-adic fractal string $\mathcal{L}_p$ by taking

$$E = B(a, p^n) \setminus \mathcal{L}_p,$$

where a is any point contained in $\mathcal{L}_p$, and n is chosen so that $B(a, p^n)$ is the smallest ball containing $\mathcal{L}_p$. Observe that $B(a, p^n)$ is a δ-neighborhood of E for an appropriately chosen δ. It follows almost immediately that

$$\begin{aligned}
\zeta_E(s) &= \int_{E_\delta} d(x,E)^{s-1}\, \mathrm{d}\mu_p(x) \\
&= \sum_{j\in\mathbb{N}} \int_{B(a_j, p^{-n_j})} d(x,E)^{s-1}\, \mathrm{d}\mu_p(x) \\
&= \sum_{j\in\mathbb{N}} (p^{-n_j})^{s-1} \int_{B(a_j, p^{-n_j})} \mathrm{d}\mu_p(x) \\
&= \sum_{j\in\mathbb{N}} \left(p^{-n_j s + n_j}\right) p^{-n_j} \\
&= \sum_{j\in\mathbb{N}} p^{-n_j s} \\
&= \zeta_{\mathcal{L}_p}(s).
\end{aligned}$$

In short, the distance zeta function simplifies to precisely the geometric zeta function defined by Lapidus and Lũ'. Placing this work in the context of this thesis resolves a couple of open questions:

Remark 4.28 ([LL09, §5.4]). Lapidus and Lũ' discuss the possibility of extending tube formulæ to the setting of non-archimedian self-similar fractal strings. Watson [Wat17] obtained tube formulæ for subsets of Ahlfors regular metric spaces. As $\mathbb{Q}_p$ is Ahlfors 1-regular, Watson's work resolves this question in the p-adic setting. Generalizations to adelic spaces or Berkovich space (see [BR10]) are still open.

Remark 4.29 ([LL09, §5.5]). Here, Lapidus and Lũ' suggest studying higher dimensional analogs of p-adic fractal strings and their associated tube formulæ. Again, the work of Watson [Wat17] fully resolves this question in the setting of $\mathbb{Q}_p^Q$. The question is unresolved in product spaces of the form

$$\prod_{j=1}^{d} X_j,$$

where either $X_j = \mathbb{Q}_{p_j}$ for some prime number p_j, or $X_j = \mathbb{R}$. This more general question is related to still-open problems regarding the complex dimensions of product spaces, and has implications in the setting of fractal subsets of adelic spaces. As discussed in Section 6.5, resolving this question is likely to be quite difficult.

Chapter 5

Local Zeta Functions and Local Dimensions

The zeta functions described in Chapter 3 can be used to give an extrinsic definition of fractality. Lapidus *et al.* [LRŽ16, p. 407] define fractality "... as the presence of at least one nonreal complex dimension," where a complex dimension is a singularity of a meromorphically extended zeta function associated to a bounded set or relative fractal drum in $\mathbb{R}^d$. The same notion extends to the context of this thesis:

Definition 5.1. A bounded set $E \subseteq (X, d, \mu)$ (or RFD (E, Ω) in (X, d, μ), respectively) is said to be *fractal* if $\mathscr{P}(\zeta_E)$ (or $\mathscr{P}(\zeta_{E,\Omega})$, respectively) contains at least one nonreal element. That is, a bounded set or RFD is fractal if it possesses at least one nonreal complex dimension.

This notion of fractality is extrinsic: a set can only be described as fractal once it has been embedded into a larger space. In this chapter, local versions of the tube and distance zeta functions are introduced. These local fractal zeta functions are defined intrinsically and do not rely on any embedding. However, the local fractal zeta functions retain many properties similar to the usual tube and distance zeta functions, and can therefore be used to give an intrinsic notion of fractality.

5.1 A brief historical note

The terms "local distance zeta function" and "local tube zeta function" are introduced by Lapidus *et al.* [LRŽ16, App. B]. In that text, the local distance zeta function associated to a bounded set $E \subseteq \mathbb{R}^d$ is the family of relative distance zeta functions associated to the RFDs (E, Ω), where Ω runs over the Borel subsets of a fixed δ-neighborhood E_δ (note that this involves a generalization the notion of an RFD to include drums taken relative to Borel sets rather than just open sets).

"Pointwise" versions of these local fractal zeta are given in equations [LRŽ16, (B.0.3),(B.0.4)], which provide definitions in terms of relative zeta functions corresponding to RFDs of the form

$$\big(\{x\}, B_{r+\delta}(x) \cap \Omega\big),$$

where r and δ are fixed positive constants, the set $B_{r+\delta}(x)$ is the annulus with inner radius r and outer radius $r + \delta$

$$B_{r+\delta}(x) = B(x, r + \delta) \setminus B(x, r),$$

and Ω is any fixed Borel set. These local zeta functions are introduced "...to consider the fractal properties of Ω near x."

The goal in this thesis is similar: in order to understand the fractal properties of a space "near" some point, introduce a zeta function which can detect those properties in some way. However, the approach presented below is distinct from that suggested by Lapidus *et al.*: the local zeta functions described in this chapter are defined for a large class of metric spaces, are not given with respect to an ambient dimension, and have convergence properties related the multifractal spectrum of a measure rather than the Minkowski dimension of a set.

5.2 Local zeta functions

In this section and the sequel, assume that $X := (X, d, \mu)$ is a metric measure space with μ a Radon measure. The intrinsic properties of a space can be studied by considering the embeddings of singleton points into that space. The first tool introduced is the local tube zeta function. This

function is related to a complexified version of the ratio used to define the lower and upper Minkowski dimensions (Definition 2.8).

Definition 5.2. Let $x \in X$ and fix a bounded open set $x \in \Omega \subseteq X$ such that $\mu(\Omega) < \infty$. The *local tube zeta function* at x relative to Ω is a function $\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) : \mathbb{C} \to \mathbb{C}$ defined by the integral

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) := \int_0^{\mathrm{diam}(\Omega)} t^{-s-1}\mu(B(x,t) \cap \Omega)\,\mathrm{d}t.$$

The local tube zeta function roughly corresponds to the relative tube zeta function associated to the RFD (x, Ω), modulo a change of sign in the exponent and the omission of a factor related to the measure theoretic Assouad dimension of the ambient space. The intuition is that the local tube zeta function is sensitive to the intrinsic geometry of the space X, and does not require an *a priori* embedding of X into a larger space.

Each local tube zeta function is related to a local distance zeta function in a manner similar to the relation between the tube and distance functions associated to a subset of a metric measure space.

Definition 5.3. Let $x \in X$ and fix a bounded open set $x \in \Omega \subseteq X$ such that $\mu(\Omega) < \infty$. The *local distance zeta function* at x relative to Ω is a function $\zeta^{\mathrm{loc}}_{x,\Omega}(s) : \mathbb{C} \to \mathbb{C}$ defined by the integral

$$\zeta^{\mathrm{loc}}_{x,\Omega}(s) := \int_\Omega d(y,x)^{-s}\,\mathrm{d}\mu(y).$$

After replacing the bound in Lemma 3.9 with the bound $\sigma < \underline{\dim}_{\mathrm{loc}}\mu(x)$, the proof carries through with minimal modification. Hence

$$\zeta^{\mathrm{loc}}_{x,\Omega}(\sigma) = \mathrm{diam}(\Omega)^{-\sigma}\mu(\Omega) + \sigma\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(\sigma) \tag{5.2.1}$$

whenever the integrals defining both functions are finite. Both the local tube and local distance zeta functions depend on a choice of a bounded open set Ω with finite measure. Once these conditions are met, the dependence on Ω is inessential with respect to the analytic properties of the zeta functions, as per the arguments leading to Lemma 3.2.

Definition 5.4. The *abscissa of* (*absolute*) *convergence* of the local distance zeta function at x is

$$D_C(\zeta^{\mathrm{loc}}_{x,\Omega}(s)) := \sup\left\{\alpha \in \mathbb{R} \;\middle|\; \int_\Omega d(y,x)^{-\alpha}\,\mathrm{d}\mu(y) < \infty\right\}.$$

Theorem 5.8 (below) shows that the abscissa of convergence coincides with the local dimension of μ at x (when it exists) and, moreover, that the integral defining the local distance zeta function converges on the half-plane to the left of the abscissa of convergence. The proof is handled in steps, via the following three lemmata, which parallel the arguments in Chapter 3 (and therefore parallel the development of the theory in [LRŽ16, §2.1]). The first lemma is a local version of the Harvey-Polking estimate (see Lemma 3.3).

Lemma 5.5. *Let $x \in X$ and fix some bounded open set $x \in \Omega \subseteq X$ with $\mu(\Omega) < \infty$. If $\sigma < \underline{\dim}_{\mathrm{loc}}\mu(x)$ then*

$$\zeta^{\mathrm{loc}}_{x,\Omega}(\sigma) = \int_\Omega d(y,x)^{-\sigma}\,\mathrm{d}\mu(y) < \infty.$$

Proof. Suppose that $\sigma \leq 0$. Then

$$d(y,x) \leq \mathrm{diam}(\Omega) \iff d(y,x)^{-\sigma} \leq \mathrm{diam}(\Omega)^{-\sigma}$$

for any $y \in \Omega$. As Ω is assumed to be bounded, it has finite diameter, and so

$$\int_\Omega d(y,x)^{-\sigma}\,\mathrm{d}\mu(y) \leq \int_\Omega \mathrm{diam}(\Omega)^{-\sigma}\,\mathrm{d}\mu(y) = \mathrm{diam}(\Omega)^{-\sigma}\mu(\Omega) < \infty,$$

where the final inequality follows from the hypothesis that Ω has finite measure. Hence the desired result holds whenever $\sigma \leq 0$.

Now suppose that $\sigma > 0$. Note that this necessarily implies that the lower local dimension of μ at x must be positive. Let Q denote this lower local dimension—that is, let

$$Q := \underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r \searrow 0} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

Fix some $\varepsilon > 0$ such that $\sigma < Q - \varepsilon$. As Ω is open, there exists some $r_1 > 0$ such that

$$B(x, r_1) \subseteq \Omega.$$

By definition of Q in terms of the limit inferior, there exists some $r_2 > 0$ such that

$$Q - \varepsilon < \frac{\log(\mu(B(x,r)))}{\log(r)}$$

whenever $r < r_2$. Fix some r^* so that

$$0 < r^* < \min\{r_1, r_2\}.$$

Hence

$$Q - \varepsilon < \frac{\log(\mu(B(x,r)))}{\log(r)} \implies \mu(B(x,r)) < r^{Q-\varepsilon} \tag{5.2.2}$$

whenever $r < r^*$.

For each $j \in \mathbb{N}$, define the annulus A_j by

$$A_j := B\big(x, 2^{-j} r^*\big) \setminus B\big(x, 2^{-(j+1)} r^*\big).$$

These annuli are disjoint and contained in Ω, and so

$$\begin{aligned} \int_\Omega d(y,x)^{-\sigma} \, \mathrm{d}\mu(y) &= \int_{\Omega \setminus B(x,r^*)} d(y,x)^{-\sigma} \, \mathrm{d}\mu(y) + \int_{B(x,r^*)} d(y,x)^{-\sigma} \, \mathrm{d}\mu(y) \\ &= \underbrace{\sum_{j=0}^{\infty} \int_{A_j} d(y,x)^{-\sigma} \, \mathrm{d}\mu(y)}_{=:I} + \underbrace{\int_{B(x,r^*)} d(y,x)^{-\sigma} \, \mathrm{d}\mu(y)}_{=:J}. \end{aligned} \tag{5.2.3}$$

As Ω has finite measure, the set $\Omega \setminus B(x, r^*)$ has finite measure as well. On this set, the distance function is bounded away from zero, and so the integral J converges absolutely in the sense of a Lebesgue. Therefore $J < \infty$.

It remains to bound I. Begin by observing that on the annulus A_j, the distance function satisfies the inequality

$$2^{-(j+1)}r^* \leq d(y,x) \leq 2^{-j}r^* \implies d(y,x)^{-\sigma} \leq 2^{\sigma(j+1)}(r^*)^{-\sigma}.$$

Substituting this into I at (5.2.3) gives

$$I = \sum_{j=0}^{\infty} \int_{A_j} d(y,x)^{-\sigma} \, \mathrm{d}\mu(y) \leq \sum_{j=0}^{\infty} 2^{\sigma(j+1)}(r^*)^{-\sigma}\mu(A_j). \tag{5.2.4}$$

The monotonicity of the measure μ combined with the estimate at (5.2.2) imply that

$$\mu(A_j) \leq \mu\left(B\left(x, 2^{-j}r^*\right)\right) \leq \left(2^{-j}r^*\right)^{Q-\varepsilon},$$

hence (5.2.4) becomes

$$\begin{aligned} I &\leq \sum_{j=0}^{\infty} 2^{\sigma(j+1)}(r^*)^{-\sigma}\mu(A_j) \\ &\leq \sum_{j=0}^{\infty} 2^{\sigma(j+1)}(r^*)^{-\sigma}\left(2^{-j}r^*\right)^{Q-\varepsilon} \\ &= 2^{\sigma}(r^*)^{Q-\sigma-\varepsilon} \sum_{j=0}^{\infty} \left(2^{\sigma-Q+\varepsilon}\right)^j. \end{aligned}$$

This final sum converges if and only if $\sigma - Q + \varepsilon < 0$. But ε was chosen so that $\sigma < Q - \varepsilon$. Therefore $I < \infty$, which completes the proof. □

Lemma 5.6. *Let $x \in X$ and fix some bounded open set $x \in \Omega \subseteq X$ with $\mu(\Omega) < \infty$. If $\sigma > \overline{\dim}_{\mathrm{loc}}\mu(x)$ then*

$$\zeta^{\mathrm{loc}}_{x,\Omega}(\sigma) = \int_{\Omega} d(y,x)^{-\sigma} \, \mathrm{d}\mu(y) = +\infty.$$

Proof. The proof of this result parallels the proof of Lemma 3.5. Fix some r^* such that $B(x, r^*) \subseteq \Omega$, and note that

$$\begin{aligned}\int_\Omega d(y,x)^{-\sigma}\,\mathrm{d}\mu(y) &= \int_{\Omega\setminus B(x,r^*)} d(y,x)^{-\sigma}\,\mathrm{d}\mu(x) + \int_{B(x,r^*)} d(y,x)^{-\sigma}\,\mathrm{d}\mu(y)\\ &\geq \int_{B(x,r^*)} d(y,x)^{-\sigma}\,\mathrm{d}\mu(y).\end{aligned}$$

It is then sufficient to show that this last integral diverges. For each $r \in (0, r^*]$, define

$$I(r) := \zeta^{\mathrm{loc}}_{x,B(x,r)}(\sigma) = \int_{B(x,r)} d(x,y)^{-\sigma}\,\mathrm{d}\mu(y).$$

As in the proof of Lemma 3.5, $I(r)$ is a nondecreasing function of r and satisfies the inequality

$$I(r) \geq r^{-\sigma}\mu(B(x,r)).$$

Let $Q = \overline{\dim}_{\mathrm{loc}}\mu(x)$ so that

$$\sigma > Q = \limsup_{r\searrow 0} \frac{\log(\mu(B(x,r)))}{\log(r)},$$

and fix some $\varepsilon \in (0, \sigma - Q)$. By definition of the limit supremum, there exists a sequence $\{r_k\}_{k=1}^\infty$ with $r_k \searrow 0$ and

$$\frac{\log(\mu(B(x,r_k)))}{\log(r_k)} \to Q \qquad \text{as } k \to \infty.$$

Then, for sufficiently large k,

$$\begin{aligned}\left|\frac{\log(\mu(B(x,r_k)))}{\log(r_k)} - Q\right| < \varepsilon \implies -\varepsilon < \frac{\log(\mu(B(x,r_k)))}{\log(r_k)} - Q < \varepsilon\\ \implies r_k^{Q-\varepsilon} > \mu(B(x,r_k)) > r_k^{Q+\varepsilon}.\end{aligned}$$

Then for any such k

$$\zeta^{\mathrm{loc}}_{x,\Omega}(\sigma) \geq I(r^*) \geq I(r_k) \geq r_k^{-\sigma}\mu(B(x,r_k)) \geq r_k^{-\sigma+Q+\varepsilon}.$$

But $-\sigma + Q + \varepsilon < 0$ by the choice of ε, and so $r_k^{-\sigma+Q+\varepsilon} \to +\infty$ as $r_k \searrow 0$, which completes the proof. □

Lemma 5.7. *Let $x \in X$, fix some bounded open set $\Omega \subseteq X$ with $\mu(\Omega) < \infty$, and suppose that there is some $s_0 \in \mathbb{C}$ such that*

$$\zeta^{\mathrm{loc}}_{x,\Omega}(s_0) = \int_\Omega |d(y,x)^{-s_0}| \, \mathrm{d}\mu(y) < \infty,$$

that is $\zeta^{\mathrm{loc}}_{x,\Omega}(s_0)$ converges absolutely as a Lebesgue integral. Then

$$\int_\Omega |d(y,x)^{-s}| \, \mathrm{d}\mu(y) < \infty$$

for any $s \in \mathbb{C}$ satisfying $\Re(s) < \Re(s_0)$.

Proof. Proof is obtained along the same lines as the proof of Lemma 3.6, *mutatis mutandis*. □

The local Harvey-Polking estimate (Lemma 5.5) together with Lemmata 5.6 and 5.7 imply that the local distance zeta function converges on a half-plane to the left of $\overline{\dim}_{\mathrm{loc}}\mu(x)$ and diverges to the right of this value. This observation is summarized by the following theorem.

Theorem 5.8. *Let $x \in X$ and suppose that $\dim_{\mathrm{loc}} \mu(x)$ exists. Then*

$$D_C := D_C(\zeta^{\mathrm{loc}}_{x,\Omega}) = \dim_{\mathrm{loc}} \mu(x).$$

and so $\zeta(s,x)$ converges for all s with $\Re(s) < D_C$, and diverges for all s with $\Re(s) > D_C$.

Definition 5.9. The *abscissa of holomorphic continuation* of the local distance zeta function at x is given by

$$D_H(\zeta^{\mathrm{loc}}_{x,\Omega}) := \sup \left\{ \alpha \in \mathbb{R} \;\middle|\; \zeta^{\mathrm{loc}}_{x,\Omega} \text{ is holomorphic on } \{\Re(s) < \alpha\} \right\}.$$

That is, $D_H(\zeta^{\mathrm{loc}}_{x,\Omega})$ is describes the boundary of the largest left half-plane to which the local distance zeta function at x may be holomorphically extended.

Lemma 5.10. *Let $x \in X$ and fix some bounded open set $x \in \Omega \subseteq X$ with $\mu(\Omega) < \infty$. The local tube zeta function $\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}$ and the local distance zeta function $\zeta^{\mathrm{loc}}_{x,\Omega}|$ are holomorphic on the open left half-plane $\{\Re(s) < \underline{\dim}_{\mathrm{loc}}\mu(x)\}$.*

Proof. Define

$$\mathrm{d}\nu(y) := \mathrm{d}\mu|_{\Omega}(y) \qquad \text{and} \qquad \varphi(y) := d(x,y),$$

where $\mu|_{\Omega}$ denotes the restriction of μ to the open set Ω. By Lemmata 5.5 and 5.7

$$\int_{\Omega} \varphi(y)^s \,\mathrm{d}\mu|_{\Omega(y)} = \int_{\Omega} d(x,y)^s \,\mathrm{d}\mu(y) = \zeta^{\mathrm{loc}}_{x,\Omega}(-s) < \infty$$

for any $s > -\underline{\dim}_{\mathrm{loc}}\mu(x)$. It then follows from Lemma 3.13 that $\zeta^{\mathrm{loc}}_{x,\Omega}$ is holomorphic on the open left half-plane $\{\Re(s) < \underline{\dim}_{\mathrm{loc}}\mu(x)\}$.

Similarly,

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(-s) = \int_0^{\delta} \varphi(t)^s \,\mathrm{d}\nu(t),$$

where

$$\mathrm{d}\nu(t) := \frac{1}{t}\,\mathrm{d}\mu|_{\Omega}(t) \qquad \text{and} \qquad \varphi(t) := t.$$

The functional relation (5.2.1) together with Lemma 5.7 imply that $\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(-s)$ converges on the right half-plane $\{\Re(s) > \underline{\dim}_{\mathrm{loc}}\mu(x)\}$, and so the desired result follows again from Lemma 3.13. □

Remark 5.11. Lemma 5.10 guarantees that both the local tube and distance zeta functions are holomorphic on the open left half-plane bounded by the lower local dimension. It therefore follows from Identity Theorem (e.g. [Sim15, Thm. 2.3.8]) that the functional relation at (5.2.1) holds on the common domain of these functions. That is,

$$\zeta^{\mathrm{loc}}_{x,\Omega}(s) = \mathrm{diam}(\Omega)^{-s}\mu(\Omega) + s\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s)$$

wherever both functions are defined.

Theorem 5.12. *Let $x \in X$, suppose that $\dim_{\text{loc}} \mu(x) = D$ exits, and fix some bounded open set $x \in \Omega \subseteq X$ with $\mu(\Omega) < \infty$. Further suppose that μ is D-homogenous on Ω. Then*

$$\lim_{\sigma \nearrow D} \tilde{\zeta}^{\text{loc}}_{x,\Omega}(\sigma) = \infty,$$

from which it follows that $D_H(\zeta^{\text{loc}}_{x,\Omega}) = D_C(\zeta^{\text{loc}}_{x,\Omega})$.

Proof. As $D = \dim_{\text{loc}} \mu(x)$, it follows from Theorem 5.8 that

$$\infty > \tilde{\zeta}^{\text{loc}}_{x,\Omega}(\sigma)$$

for all $\sigma < D$. It therefore "makes sense" to consider the limit of $\tilde{\zeta}^{\text{loc}}_{x,\Omega}(\sigma)$ as σ increases to the local dimension of the measure at x.

Fix some $\delta \in (0, 1)$ such that $B(x, \delta) \subseteq \Omega$. By hypothesis, the restriction of μ to Ω is D-homogenous, and so there is some $M > 0$ such that

$$\frac{\mu(B(x,\delta))}{\mu(B(x,r))} \leq M \left(\frac{\delta}{r}\right)^D \implies \frac{\mu(B(x,\delta))}{M\delta^D} r^D \leq \mu(B(x,r))$$

for all $0 < r < \delta$. Combining the constants into a single term, this implies that there is $C > 0$ such that

$$\mu(B(x,r)) \geq C r^D$$

for all $r < \delta$. Hence

$$\begin{aligned}
\lim_{\sigma \nearrow D} \tilde{\zeta}^{\text{loc}}_{x,\Omega}(\sigma) &= \int_0^{\text{diam}(\Omega)} t^{-\sigma-1} \mu(B(x,t) \cap \Omega)\, \mathrm{d}t \\
&\geq \lim_{\sigma \nearrow D} \int_0^{\delta} t^{-\sigma-1} \mu(B(x,t))\, \mathrm{d}t \\
&\geq C \lim_{\sigma \nearrow D} \int_0^{\delta} t^{-\sigma-1} t^D\, \mathrm{d}t \\
&= C \lim_{\sigma \nearrow D} \frac{\delta^{D-\sigma}}{D-\sigma} \\
&= \infty.
\end{aligned}$$

Thus the local tube zeta function possesses a singularity at $\underline{\dim}_{\text{loc}}\mu(x)$, which implies that

$$D_H(\zeta^{\text{loc}}_{x,\Omega}) \leq D_C(\zeta^{\text{loc}}_{x,\Omega}).$$

Equality follows from Lemma 5.10. □

Definition 5.13. Let $x \in X$ and fix an appropriate neighborhood Ω of x. Suppose that $\zeta^{\text{loc}}_{x,\Omega}(s)$ admits a meromorphic extension to some open domain W containing the critical line $\{\Re(s) = D(\zeta^{\text{loc}}_{x,\Omega})\}$. The set

$$\mathscr{P}(\zeta^{\text{loc}}_{x,\Omega}) = \left\{\omega \;\middle|\; \omega \in \mathscr{P}_U(\zeta^{\text{loc}}_{x,\Omega}) \text{ for some } U\right\}$$

is the *set of local complex dimensions* of X at x. The notation here is consistent with that in Definition 3.19.

The local complex dimensions may also be understood in terms of the poles of a meromorphic continuation of the local tube zeta function. However, the functional equation (5.2.1) implies that the local tube zeta function may have an additional pole of order one at zero.

5.3 Examples

Example 5.14 (A point in $\mathbb{Q}_p$)**.** In Example 4.20, it was observed that a singleton point in $\mathbb{Q}_p$ exhibits geometric oscillation, in the sense that it has nonreal complex dimensions. Per Definition 5.1, this result is interpreted to mean that a singleton point in $\mathbb{Q}_p$ is a fractal.

An alternative interpretation is that the complex dimensions of this singleton point provide information about the ambient space of p-adic numbers. It is not that the point is fractal, but rather that it has been embedded into a fractal space. This is a conclusion which can be phrased in terms of local complex dimensions.

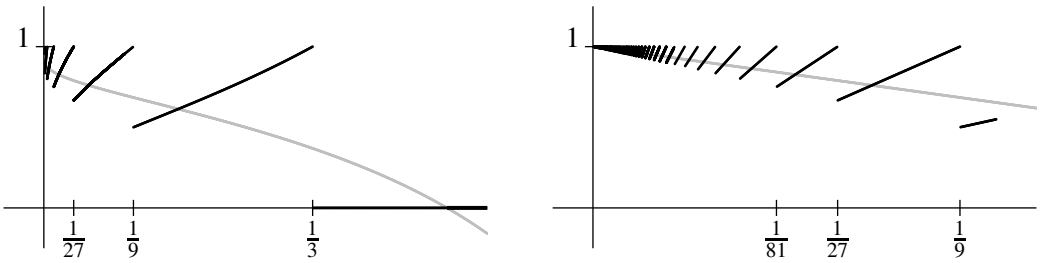


Figure 5.1: In both graphs, the black lines are the graph of the function

$$r \mapsto \frac{\log(\mu_3(B(x,r)))}{\log(r)},$$

where $x \in \mathbb{Q}_p$ and $r > 0$, and the grey line is the graph of the corresponding function

$$r \mapsto \frac{\log(m(B(\xi,r)))}{\log(r)},$$

where $\xi \in \mathbb{R}$, $r > 0$, and m is the usual one-dimensional Lebesgue measure on r. The graph on the right has been log-scaled. In both the real and 3-adic cases, the ratio tends to 1 as r goes to zero, which corresponds to the fact that

$$\dim_{\text{loc}} \mu_3(x) = \dim_{\text{loc}} m(\xi) = 1$$

for any $x \in \mathbb{Q}_3$ and $\xi \in \mathbb{R}$.

Paralleling Example 4.20, let $x = 0$ and fix $\Omega = \mathbb{Z}_p$. Via a computation that is substantially similar to that in Example 4.20, the local distance zeta function corresponding to this point is then

$$\zeta^{\text{loc}}_{0,\mathbb{Z}_p}(s) = \int_{\mathbb{Z}_p} d(x,0)^{-s} \, \mathrm{d}\mu_p(x) = \frac{p-1}{p}\left(\frac{1}{1-p^{s-1}}\right).$$

The local distance tube zeta function extends to a mentire function which possesses a simple pole whenever

$$s \in 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)} = \mathscr{P}(\zeta^{\text{loc}}_{0,\mathbb{Z}_p}).$$

By a translation, a similar result will hold for any $x \in \mathbb{Q}_p$. That is, if $x \in \mathbb{Q}_p$, then

$$\mathscr{P}(\zeta^{\text{loc}}_{x,\mathbb{Z}_p}) = 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}.$$

These local complex dimensions lie on a vertical line with real part 1, which comports with the observation that $\dim_{\text{loc}} \mu(x) = 1$ for all $x \in \mathbb{Q}_p$. Moreover, the nonreal complex dimensions indicate that the measure μ_p exhibits an oscillatory behaviour around each point. This behaviour can be seen in Figure 5.1, which plots the ratio

$$\frac{\log(\mu_3(B_{\leq}(x,r)))}{\log(r)}$$

against the radius r. The behaviour is similar in $\mathbb{Q}_p$ for any prime p, though the oscillations are of higher frequency for larger p. ◁

Example 5.15 (The ternary Cantor set)**.** Let $(\Phi, \mathfrak{p}) = \{(\varphi_0, p_0), (\varphi_1, p_1)\}$ be the weighted SSIFS with maps

$$\varphi_0(x) = \frac{1}{3}x \qquad \text{and} \qquad \varphi_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

and weights $\mathfrak{p}_0 = \mathfrak{p}_1 = \frac{1}{2}$. Let $\mathscr{C}$ denote the attractor of Φ, and let μ denote the induced self-similar measure supported on $\mathscr{C}$. As a set, $\mathscr{C}$ is the "usual" ternary Cantor set. This set is complete with respect to the subspace metric d induced by the Euclidean metric on $\mathbb{R}$, and μ coincides with the $\log_3(2)$-dimensional Hausdorff measure supported on $\mathscr{C}$.

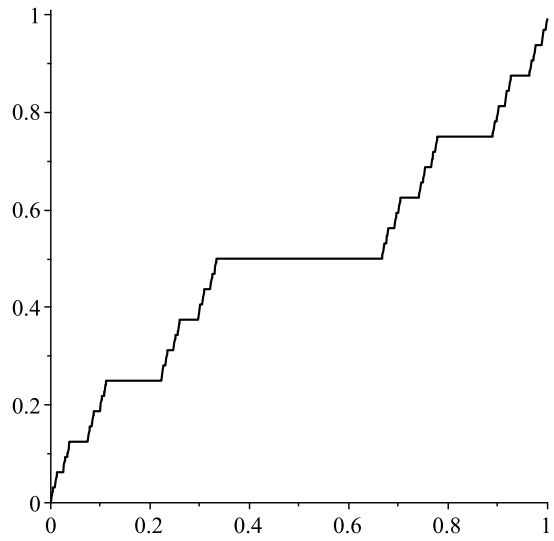


Figure 5.2: A graph of k, the Cantor function, on the interval $[0, 1]$. If t is an element of $[0, \infty)$, then $\mu(B(0, t)) = k(x)$.

Take $x = 0$, $\Omega = B(0, 1)$, and let $k : \mathbb{R} \to \mathbb{R}$ denote the Cantor function, pictured in Figure 5.2. The local tube zeta function at x relative to Ω is given by

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = \int_0^1 t^{-s-1} k(t)\,\mathrm{d}t = \int_0^{1/3} t^{-s-1} k(t)\,\mathrm{d}t + \underbrace{\int_{1/3}^1 t^{-s-1} k(t)\,\mathrm{d}t}_{=:\vartheta(s)}, \tag{5.3.1}$$

where ϑ is an entire function. For any $t \in [0, 1]$, the Cantor function satisfies the relation

$$k\left(\frac{t}{3}\right) = \frac{1}{2} k(t).$$

Thus, making the change of variables $\tau = 3t$,

$$\int_0^{1/3} t^{-s-1} k(t)\,\mathrm{d}t = \int_0^1 \left(\frac{\tau}{3}\right)^{-s-1} k\left(\frac{\tau}{3}\right) \frac{\mathrm{d}\tau}{3} = \frac{3^s}{2} \zeta^{\mathrm{loc}}_{x,\Omega}(s).$$

Substituting this into (5.3.1) and solving for the local tube zeta function gives

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = \frac{3^s}{2}\zeta^{\mathrm{loc}}_{x,\Omega}(s) + \vartheta(s) \implies \tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = \frac{2}{2-3^s}\vartheta(s).$$

The first term has poles whenever $3^s = 2$, that is for all

$$s \in \omega_k = \frac{\log(2)}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}.$$

Each one of these poles is a pole of the local tube zeta function as long as $\vartheta(\omega_k) \neq 0$. While computing ϑ explicitly is quite difficult, it can be bounded away from zero near ω_0. If $s \in (0,1)$, then

$$\begin{aligned}
\vartheta(s) &= \int_{1/3}^{1} t^{-s-1}k(t)\,\mathrm{d}t \\
&\geq \int_{1/3}^{1} k(t)\,\mathrm{d}t && (s+1>1 \text{ and } t\leq 1, \text{ so } t^{-s-1}>1) \\
&\geq \frac{1}{2}\int_{1/3}^{1} \mathrm{d}t && (k(t)\geq 1/2 \text{ for } t\in[1/3,1]) \\
&= \frac{1}{3}.
\end{aligned}$$

Thus on the interval $(0,1)$, the function ϑ is bounded away from zero. In particular, no cancelation occurs at $s = \log(2)/\log(3)$, which implies that

$$\lim_{s\to\log_3(2)} \tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = +\infty.$$

That is, ω_0 is a pole of the local tube zeta function.

If x is any rational point of the Cantor set—that is, if there is a word $\boldsymbol{i} \in \mathscr{I}^*$ such that $x \in \varphi_{\boldsymbol{i}}(\{0,1\})$—then a similar argument holds by considering $\Omega = B(x, 3^{|\boldsymbol{i}|})$. That is, for any rational points $x \in \mathscr{C}$, the local tube zeta function relative to $B(x, 3^{|\boldsymbol{i}|})$ will be as above, up to some scaling. The structure of the local complex dimensions of $\mathscr{C}$ at irrational points is not currently

understood, though it is conjectured that it is identical to the structure at rational points, as the Cantor set is quite homogeneous. ◁

In Example 5.15, the self-similar structure of the Cantor set and the regularity of the natural Hausdorff measure supported on that set make it possible to obtain a tractable representation of the local tube zeta function. In light of the work done by Watson [Wat17], this example is not surprising: the usual $\log_3(2)$-dimensional Hausdorff measure is Ahlfors regular on the Cantor set. However, similar techniques to those used in Example 5.15 may be applied to recover information about the multifractal spectrum of measures which are not Ahlfors regular, where previous results do not apply. For instance, Example 5.16 describes a family of irregular self-similar measures supported on the Cantor set.

Example 5.16 (A family of self-similar measures on the Cantor set)**.** Let $(\Phi, \mathfrak{p}) = \{(\varphi_0, \mathfrak{p}_0), (\varphi_1, \mathfrak{p}_1)\}$ be the weighted SSIFS with maps

$$\varphi_0(x) = \frac{1}{3}x \qquad \text{and} \qquad \varphi_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

and weights $\mathfrak{p}_0 = \mathfrak{q}$ and $p_1 = 1-\mathfrak{q}$, where $\mathfrak{q} \in (0,1)$. Let $\mathscr{C}$ denote the attractor of Φ, and let $\mu_{\mathfrak{p}}$ denote the induced self-similar measure supported on $\mathscr{C}$. As in the previous example, the underlying metric space is the usual ternary Cantor set with the subspace metric inherited from $\mathbb{R}$. For $\mathfrak{q} \notin \{0, 1/2, 1\}$, the measure $\mu_{\mathfrak{p}}$ is an irregular (in the sense of Ahlfors) multifractal measure.

The local tube zeta function at 0 relative to $B(0,1)$ is

$$\tilde{\zeta}^{\mathrm{loc}}_{0,B(0,1)}(s) = \int_0^1 t^{-s-1}\mu_{\mathfrak{p}}(B(0,t))\,\mathrm{d}t = \int_0^{1/3} t^{-s-1}\mu_{\mathfrak{p}}(B(0,t))\,\mathrm{d}t + \underbrace{\int_{1/3}^1 t^{-s-1}\mu_{\mathfrak{p}}(B(0,t))\,\mathrm{d}t}_{=:\vartheta(s)},$$

where ϑ is an entire function. For any $t \in [0,1]$, the self-similarity of the measure gives

$$\mu_{\mathfrak{p}}\left(B\left(0, \frac{t}{3}\right)\right) = \mathfrak{q}\mu_{\mathfrak{p}}(B(0,t)).$$

Make the change of variables $\tau = 3t$ to obtain

$$\begin{aligned}\tilde{\zeta}^{\mathrm{loc}}_{0,B(0,1)}(s) &= \int_0^1 \left(\frac{\tau}{3}\right)^{-s-1} \mu_{\mathfrak{p}}\left(B\left(0,\frac{t}{3}\right)\right)\frac{\mathrm{d}\tau}{3} + \vartheta(s)\\ &= \mathfrak{q}3^s \int_0^1 \tau^{-s-1}\mu_{\mathfrak{p}}(B(0,\tau))\,\mathrm{d}\tau + \vartheta(s)\\ &= \mathfrak{q}3^s\tilde{\zeta}^{\mathrm{loc}}_{0,B(0,1)}(s) + \vartheta(s).\end{aligned}$$

Solve for the local tube zeta function to get

$$\tilde{\zeta}_{0,B(0,1)}(s) = \frac{\vartheta(s)}{1-\mathfrak{q}3^s}. \tag{5.3.2}$$

The denominator is zero when

$$s \in \frac{\log(1/\mathfrak{q})}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}, \tag{5.3.3}$$

and, by arguments similar to those in the previous example, ϑ is bounded away from zero when $s \in (0,1)$. Therefore the local tube zeta function possesses a simple pole at

$$s = \frac{\log(1/\mathfrak{q})}{\log(3)}.$$

This corresponds to the local dimension of $\mu_{\mathfrak{p}}$ at 0. Details of the computation of the local dimension are given in a more general setting in Proposition 5.17.

Similar arguments demonstrate that

$$\tilde{\zeta}_{1,B(1,1)}(s) = \frac{\vartheta(s)}{1-(1-\mathfrak{q})3^s},$$

where ϑ is an entire function (though not the same function as in (5.3.2)). This possesses a simple pole at

$$s = \frac{\log(1/(1-\mathfrak{q}))}{\log(3)},$$

which corresponds to $\dim_{\mathrm{loc}}\mu(1)$. ◁

More generally, if a self-similar measure arises from a weighted iterated function system which satisfies the open set condition, then similar arguments apply.

Proposition 5.17. *Let $(\Phi, \mathfrak{p})$ be a weighted self-similar iterated function system on $\mathbb{R}^d$ which satisfies the open set condition. Let $\varphi \in \Phi$ be one of the contracting similitudes, let c denote the contraction ratio of φ, and let $\mathfrak{q} \in \mathfrak{p}$ denote the weight associated to φ.*

Take x to be the unique fixed point of φ, that is, x is the unique point such that

$$\varphi(x) = x.$$

Choose $\Omega = B(x, r)$ so that

$$\varphi(B(x, r)) \cap \psi(B(x, r)) = \varnothing$$

for any $\psi \in \Phi$ with $\psi \neq \varphi$ (such a choice is possible, as Φ satisfies the open set condition). Finally, let $\mu_{\mathfrak{p}}$ denote the self-similar measure supported on the attractor of Φ which is induced by $(\Phi, \mathfrak{p})$.

The local tube zeta function at x relative to Ω is

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = \frac{\vartheta(s)}{1 - \mathfrak{q}c^{-s}}$$

where ϑ is an entire function. Moreover, $\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}$ possesses a simple pole on the real axis at $\dim_{\mathrm{loc}} \mu(x)$.

Proof. The local tube zeta function is given by

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = \int_0^r t^{-s-1} \mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t = \int_0^{cr} t^{-s-1} \mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t + \underbrace{\int_{cr}^r t^{-s-1} \mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t}_{=:\vartheta(s)}, \tag{5.3.4}$$

where ϑ is an entire function. By hypothesis, Φ satisfies the open set condition for an open set U, and Ω is contained in U. Thus

$$\varphi(\Omega) \cap \psi(\Omega), = \varnothing$$

where ψ any map in Φ other than φ. The self-similar structure of the measure then gives

$$\mu(B(x,t)) = \mathfrak{q}\mu(B(x,c^{-1}t))$$

for any $t \in [0,cr]$. After the change of variables $t = c\tau$, the self-similarity of the measure implies that

$$\int_0^{cr} t^{-s-1}\mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t = \mathfrak{q}c^{-s}\int_0^{r} t^{-s-1}\mu_{\mathfrak{p}}(B(x,\tau))\,\mathrm{d}\tau = \mathfrak{q}c^{-s}\tilde{\zeta}_{x,B(x,r)}(s).$$

Substitute this into (5.3.4) and isolate the local tube zeta function to obtain

$$\tilde{\zeta}_{x,B(x,r)}(s) = \mathfrak{q}c^{-s}\tilde{\zeta}_{x,B(x,r)}(s) + \vartheta(s) \implies \tilde{\zeta}_{x,B(x,r)}(s) = \frac{\vartheta(s)}{1-\mathfrak{q}c^{-s}}.$$

As ϑ is entire, the set of local complex dimensions is contained in the zero set of the denominator, that is

$$\mathscr{P}(\zeta_{x,B(x,r)}) \subseteq \frac{\log(\mathfrak{q})}{\log(c)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(c)},$$

which is the claimed form of the local tube zeta function. For s on the positive real axis, $\vartheta(s)$ is bounded away from zero: if $s > 0$, then

$$\begin{aligned}
\vartheta(s) &= \int_{cr}^{r} t^{-s-1}\mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t \\
&\geq r^{-s-1}\int_{cr}^{r}\mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t && (t\in[cr,r] \text{ and } -s-1<0) \\
&\geq r^{-s-1}(r-cr)\mu_{\mathfrak{p}}(B(x,cr)) \\
&= \frac{1-c}{r^s}\mu_{\mathfrak{p}}(B(x,cr)).
\end{aligned}$$

The self-similar measure supported on $\mathscr{A}$ gives positive measure to every ball of positive radius centered in $\mathscr{A}$, hence $\mu_{\mathfrak{p}}(B(x,cr)) > 0$. Therefore

$$\vartheta(s) \geq \frac{1-c}{r^s}\mu_{\mathfrak{p}}(B(x,cr)) > 0$$

for all $s \in (0, \infty)$. In particular,

$$\vartheta\left(\frac{\log(\mathfrak{q})}{\log(c)}\right) \neq 0,$$

which implies that $\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}$ has a simple pole at

$$\frac{\log(\mathfrak{q})}{\log(c)}.$$

Observe that if $\rho \in (0, r)$, then there exists some natural number n such that

$$c^{n+1} r \leq \rho < c^n r. \tag{5.3.5}$$

Via the self-similar structure of $\mu_{\mathfrak{p}}$,

$$\mu(B(x, c^n r)) = \mathfrak{q}^n C,$$

where $C = \mu(B(x, r))$ is a constant. Hence

$$\begin{aligned}(n+1)\log(\mathfrak{q}) + \log(C) &= \log(\mu(B(x, c^{n+1} r))) \\ &\leq \log(\mu(B(x, \rho))) \\ &\leq \log(\mu(B(x, c^n r))) \\ &= n\log(\mathfrak{q}) + \log(C).\end{aligned}$$

Divide through by $\log(\rho)$ and use the estimate (5.3.5) to obtain

$$\frac{(n+1)\log(p) + \log(C)}{(n+1)\log(c)} \geq \frac{\log(\mu(B(x, \rho)))}{\log(\rho)} \geq \frac{n\log(\mathfrak{q}) + \log(C)}{n\log(c)}.$$

As ρ tends to zero, n tends to infinity, thus by the squeeze theorem

$$\begin{aligned}\frac{\log(\mathfrak{q})}{\log(c)} &= \lim_{n\to\infty} \frac{(n+1)\log(\mathfrak{q}) + \log(C)}{(n+1)\log(c)} \\ &\geq \lim_{\rho\to 0} \frac{\log(\mu(B(x,\rho)))}{\log(\rho)} \\ &\geq \lim_{n\to\infty} \frac{n\log(\mathfrak{q}) + \log(C)}{n\log(c)} \\ &= \frac{\log(\mathfrak{q})}{\log(c)}.\end{aligned} \tag{5.3.6}$$

The limit (5.3.6) therefore exists, and is the local dimension of the measure. That is,

$$\dim_{\mathrm{loc}}\, \mu(x) = \frac{\log(\mathfrak{q})}{\log(c)}.$$

Note that Example 5.16 corresponds to the case $c = 1/3$. □

Corollary 5.18. *Let $(\Phi, \mathfrak{p})$ be a weighted self-similar iterated function system which satisfies the open set condition for an open set U. Fix $\boldsymbol{i} \in \mathscr{I}$, let $x_{\boldsymbol{i}}$ be the fixed point of $\varphi_{\boldsymbol{i}}$, and let $c_{\boldsymbol{i}}$ denote the contraction ratio of $\varphi_{\boldsymbol{i}}$, i.e.*

$$c_{\boldsymbol{i}} = \prod_{k=1}^{|\boldsymbol{i}|} c_{i_k}.$$

Choose $\Omega = B(x_{\boldsymbol{i}}, r)$ so that

$$\Omega \subseteq U.$$

The local tube zeta function at x relative to Ω is

$$\tilde{\zeta}^{\mathrm{loc}}_{x_{\boldsymbol{i}},\Omega}(s) = \frac{\vartheta(s)}{1 - \mathfrak{p}_{\boldsymbol{i}} c_{\boldsymbol{i}}^{-s}},$$

where ϑ is an entire function. Moreover, $\tilde{\zeta}^{\mathrm{loc}}_{x_{\boldsymbol{i}},\Omega}$ possess simple pole at $\dim_{\mathrm{loc}}\, \mu(x)$.

Proof. The collection of maps

$$\Psi = \{\varphi_{\boldsymbol{j}} \mid j \in \mathscr{I}^*, |\boldsymbol{j}| = |\boldsymbol{i}|\}$$

is a self-similar iterated function system which has the same attractor as Φ. Note that the contraction ratio of $\varphi_{\boldsymbol{j}}$ is $c_{\boldsymbol{j}}$. To each map $\varphi_{\boldsymbol{j}} \in \Psi$, associate the weight

$$\mathfrak{p}_{\boldsymbol{j}} = \prod_{k=1}^{|\boldsymbol{j}|} \mathfrak{p}_{j_k}$$

The self-similar measure induced by Ψ with these weights is the same as that induced by $(\Phi, \mathfrak{p})$. The conclusion follows by application of Proposition 5.17. □

The preceding proposition and corollary demonstrate how the local tube zeta function recovers the local dimension of certain self-similar measures supported on the Cantor set. This falls under the umbrella of "fine multifractal analysis" (see, for example, [Fal04, Ch. 17]). A natural next step is to consider the properties of sets which exhibit the same "local structure". Traditionally, this means studying sets on which the measure exhibits some fixed scaling property, i.e.,

$$X_\alpha = \{x \in X \mid \dim_{\mathrm{loc}} \mu(x) = \alpha\}.$$

Roughly speaking, the fine multifractal spectrum is a function of the scaling exponents, given by

$$\alpha \mapsto \dim(X_\alpha),$$

where dim is fixed notion of dimension (typically the Hausdorff dimension).

In the case of an inhomogeneous self-similar measure, e.g. the measure $\mu_{\mathfrak{p}}$ in Example 5.16 for any $\mathfrak{q} \notin \{0, 1/2, 1\}$, the set X_α will "typically" have fractal structure. As α varies, different fractals are described. This observation leads to the terminology "multifractal measure".

In the context of the local complex dimensions of a multifractal measure, this idea might be expanded to the examination of sets which possess the same local complex dimensions, that is, sets of the form

$$X_{\mathscr{P}} = \left\{ x \in X \;\middle|\; \mathscr{P}(\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}) = \mathscr{P} \right\},$$

where $\mathscr{P}$ is some fixed set of points in $\mathbb{C}$.

Other classes of zeta functions have been introduced in the past to study the properties of multifractal measures. Rock's thesis [Roc17] introduced a family of multifractal zeta functions to study measures subordinate to certain one-dimensional fractal harps. Subsequent works have expanded on the key ideas introduced by Rock: Lapidus and various collaborators [LR09, Lap09, dSLRR13, ELMR15] have studied multifractal mesures on $\mathbb{R}$, while Véhel and Mendivil expand the approach to higher dimensional Euclidean spaces [JLVFM10].

The emphasis of these previous works is on the "course multifractal analysis" (again, see [Fal04, Ch. 17]), which gives global information about the oscillations or fluctuations of a measure at any particular scale, but which gives no information about the behaviour of a measure at a point. Proposition 5.17 and its corollary suggest that the local fractal zeta functions are promising complementary tools, as they give information about the oscillations of a measure on a pointwise basis.

Chapter 6

Future Directions

6.1 Self-similar measures

When $\mu_{\mathfrak{p}}$ is a self-similar measure corresponding to a SSIFS which satisfies the open set condition, Proposition 5.17 gives an explicit computation of the principle (leftmost real) local complex dimension of a fixed point of any of the maps composing the IFS. In the setting of this proposition, the local tube zeta function is given by

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) = \frac{\vartheta(s)}{1 - \mathfrak{q}c^{-s}},$$

where ϑ is an entire function, explicitly given on an open left half-plane by the integral

$$\int_{cr}^{r} t^{-s-1}\mu_{\mathfrak{p}}(B(x,t))\,\mathrm{d}t.$$

Intuition suggests that ϑ should not vanish on the set

$$\{s \in \mathbb{C} \mid 1 - \mathfrak{q}c^{-s}\},$$

and that the complex dimensions should be precisely

$$\mathscr{P}(\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}) = \frac{\log(\mathfrak{q})}{\log(c)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(c)}.$$

Verification of this intuition is currently open.

Question 6.1. *In the setting of Proposition 5.17, can it be shown that*

$$|\vartheta(\omega_k)| = \left| \int_{cr}^{r} t^{-\omega_k - 1} \mu_{\mathfrak{p}}(B(x,t))\, \mathrm{d}t \right| > 0$$

for all $k \in \mathbb{Z}$*, where*

$$\omega_k = \frac{\log(\mathfrak{q})}{\log(c)} + \mathbb{i}\frac{2\pi k}{\log(c)}?$$

Corollary 5.18 extends the result of Proposition 5.17 to the "rational points" of the attractor of Φ. It should be possible to extend the result to the remaining points of the attractor, but it is not clear that a "naive" limiting argument is possible.

Question 6.2. *Can the approach outlined in Corollary 5.18 and a limiting argument be combined in order to obtain the full (fine) multifractal spectrum of the self-similar measure induced by a weighted iterated function system satisfying the open set condition?*

Finally, Proposition 5.17 and Corollary 5.18 depend on the open set condition in order to precisely determine the measures of certain balls relative to their preimages under the mappings of the iterated function system. This condition is required in order to ensure that there is a neighborhood of x which has non-intersecting images under the IFS.

The open set condition is one of several separation conditions which have been studied in the literature on self-similar and self-affine sets. Other separation conditions include the weak separation property (which gives control over the monoidal structure of the system of functions; see [LN99, Zer96]) and the finite type condition (which gives control over the number of image balls which may appear in a bounded region; see [NW01]).

Question 6.3. *Can results parallel to those in Proposition 5.17 and Corollary 5.18 be obtained if the open set condition is replaced with a weaker separation condition?*

6.2 Spectrum and Geometry

The fractal zeta functions were originally introduced to study the Laplace operator on fractal harps, i.e. bounded open subsets of the real line. In this setting, the geometric and spectral zeta functions provide a language for describing the link between the geometry of a fractal harp and the spectrum of the corresponding operator. For further discussion, see [Lap93, LvF13].

The connection between geometry and spectrum is less clear in higher dimensional Euclidean settings. A brief overview of what is known is described by Lapidus *et al.* [LRŽ16, §4.3.1]. In more general metric spaces, the picture is even cloudier, though some results have been obtained. For example, Lehrbäck and Tuominen [LT13] establish Hardy-type inequalities in certain homogeneous metric spaces via fractal zeta functions.

A potential obstruction in the investigation of the relation between spectrum and geometry is that many of the important tools used in the study of analysis and differential operators are essentially local—continuity, differentiability, and homogeneity are all local properties, while many of the tools used in fractal geometry—such as most notions of dimension—are essentially global.

Question 6.4. *Can local fractal zeta functions be used to detect local "pathologies" in spaces? For example, finding solutions to differential equations is greatly complicated when the boundary of the domain has cusps. How do the local fractal zeta functions see such cusps? Is there any relation between the local complex dimensions at a cusp point and the spectrum of an operator acting on the domain?*

Related questions concern the behaviour of systems in domains with fractal boundary. For example, Lapidus and Pang [LP95] observed that there are sequences of points along the boundary of the von Koch domain (the open set in $\mathbb{R}^2$ bounded by the classical von Koch snowflake) along which the magnitude of the gradient increases without bound—these sequences are related to "twist

points" of the von Koch curve, which are discussed by Di Biase *et al.* [DBFU98]. These results are explored numerically and visualized by Lapidus *et al.* [LNRG96].

Question 6.5. *Is there a relation between the behaviour of local fractal zeta functions at twist points and the behaviour of eigenfunctions the Laplace equation with Dirichlet conditions on the von Koch domain?*

6.3 Embedding problems

What is now known as the Assouad dimension was first studied in the early 20th Century by Minkowski and Bouligand. At the time, it received little attention, as it lacks many of the desirable properties of other notions of dimension, such as countable stability or bounds on the dimension of Cartesian products in terms of the dimension of the factors. The notion was reintroduced by Assouad [Ass79] in relation to certain embedding problems: he shows that the image of a homogenous set under a bi-Lipschitz map is homogenous, and under certain conditions, a homogenous subset of an infinite-dimensional Hilbert space may be embedded into a finite dimensional Euclidean space via a bi-Lipschitz map.

Related embedding results have been of interest over the last half-century. For example, suppose that E is a subset of some Hilbert space and that

$$\dim_{\mathrm{H}}(E - E) \leq d < \infty,$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension and $E - E$ is the Minkowski difference, that is,

$$E - E = \{x - y \mid x, y \in E\}.$$

Mañé [Mañ81] proves that, under these hypotheses, a residual (comeager or of the second category, in the sense of Baire) set of projections continuously embed E into $\mathbb{R}^d$. In other words, if the set of differences $E - E$ is sufficiently regular, then E can be embedded into a finite dimensional Euclidean space by a larger family of projections.

Other more recent results follow along similar lines. Hunt and Kaloshin [HK99] show that if $\overline{\dim}_{\mathrm{Mi}}(E-E) < d$, then a prevalent set of projections continuously embed E into $\mathbb{R}^D$ with Hölder continuous inverse, where prevalence is a notion of density introduced by Hunt, Saur, and Yorke [HSY92]. Olson and Robinson [OR10] prove that if $\dim_{\mathrm{As}}(E-E) < d$, then a prevalent set of projections continuously embed E into $\mathbb{R}^d$ with Lipschitz inverse, up to a logarithmic correction. Essentially, the means that if E is the infinite dimensional attractor of a dynamical system, but $\dim(E-E)$ is finite (where dim is a suitable notion of dimension), then E may be embedded into a finite dimensional space while preserving the dynamics of the system. Similar results are given in [Ols02, Theorem 5.2] and [Rob11, Theorem 9.20].

A theme among these results is that the embeddability of a space E depends not on the dimension of E itself, but on the Minkowski difference $E-E$. This suggests that spaces which are not embeddable in a "nice" way may have additional structure—such as lower-dimensional geometric oscillation—which is not being seen by traditional notions of dimension.

Question 6.6. *Fractal zeta functions are more sensitive to various geometric structures than traditional notions of dimension. Can the complex dimensions of a space be used to refine or restate embedding results such those described above? Can these embedding results be stated in terms of properties of the spaces themselves, rather than their Minkowski differences?*

6.4 Refining bounds on the abscissæ of convergence

Let (X, d, μ) be a metric space with μ a Radon measure, let $x \in X$, and fix a bounded open $\Omega \subseteq X$ with $\mu(\Omega) < \infty$. In Section 5.2, it is shown that the integral defining the local distance zeta function $\zeta^{\mathrm{loc}}_{x,\Omega}$ is absolutely convergent (in the sense of a Lebesgue integral) on a half-plane to the left of the lower local dimension of the measure μ at x. It is also shown that the defining integral diverges on a half-plane to the right of the upper local dimension of the measure μ at x.

If the upper and lower local dimensions of μ coincide at x, then the abscissa of convergence will be the common value. In the case that the two do not coincide, then the location of the abscissa of convergence is currently unknown.

Question 6.7. *If (X, d, μ) is an appropriate metric measure space and $x \in X$ is such that*

$$\underline{\dim}_{\mathrm{loc}}\mu(x) < \overline{\dim}_{\mathrm{loc}}\mu(x)$$

(*that is, the local lower dimension of μ at x is* strictly *less than the upper local dimension*)*, what more can be said about the abscissa of convergence of the local fractal zeta functions $\zeta_{x,\Omega}^{\mathrm{loc}}$ and $\tilde{\zeta}_{x,\Omega}^{\mathrm{loc}}$? What are the necessary and sufficient hypotheses under which the abscissa of convergence will coincide with either the upper or lower local dimension of μ at x?*

A similar question may be asked about the (global) fractal zeta functions ζ_E and $\tilde{\zeta}_E$: under the hypotheses that the Minkowski dimension of E exists and is strictly smaller than the ambient dimension, and that E has lower Minkowski content in that dimension, Theorem 3.17 asserts that the abscissa of convergence will coincide with the abscissa of holomorphic continuation.

Question 6.8. *If the hypotheses of Theorem 3.17 are weakened, what can be said? Are there any examples of a set E for which $\dim_{\mathrm{Mi}}(E) = D$ exists and $\underline{\mathfrak{M}}^D(E) = 0$ such that ζ_E extends to a holomorphic function on a domain containing D? What are the necessary and sufficient hypotheses under which the abscissæ of convergence and holomorphic continuation will coincide?*

6.5 The non-archimedean $(3, 5)$-adic Cantor dust

This section examines a "self-affine" subset of $\mathbb{Q}_3 \times \mathbb{Q}_5$ with respect to the L_∞ metric, i.e. that given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|_3, |x_2 - y_2|_5\},$$

where $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, with $x_1, y_1 \in \mathbb{Q}_3$ and $x_2, y_2 \in \mathbb{Q}_5$. The measure on this space is the natural product measure, denoted by μ. With respect to this metric and measure, $\dim_{\mathrm{Me}}(\mathbb{Q}_3 \times \mathbb{Q}_5) = 2$.

Informally, the goal is to consider a product of Cantor-like sets: one in $\mathbb{Q}_3$ with contraction ratio $1/3$, and one in $\mathbb{Q}_5$ with contraction ratio $1/5$. This set can be realized as the attractor of a self-affine IFS consisting of four maps that take $\mathbb{Z}_3 \times \mathbb{Z}_5$ into the four "corners" of $\mathbb{Z}_3 \times \mathbb{Z}_5$. More

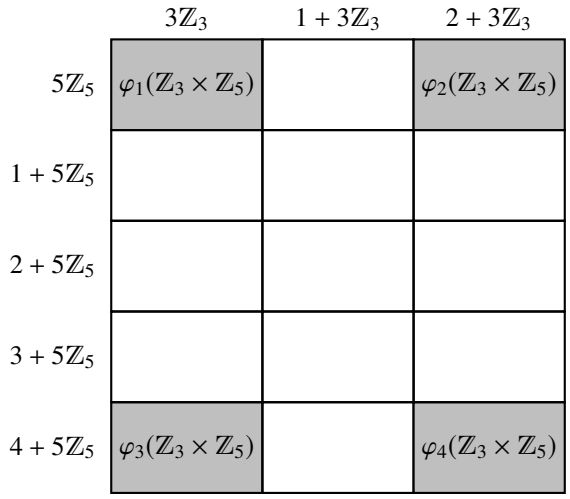


Figure 6.1: A diagrammatic depiction of the action of a the maps $\{\varphi_j\}_{j=1}^4$ on the set $\mathbb{Z}_3 \times \mathbb{Z}_5 \subseteq \mathbb{Q}_3 \times \mathbb{Q}_5$. Note that this IFS satisfies the open set condition with $U = \mathbb{Z}_3 \times \mathbb{Z}_5$.

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precisely, let $\mathscr{A}$ be the self-affine attractor of the system given by the four contraction mappings $\varphi_j : \mathbb{Q}_3 \times \mathbb{Q}_5 \to \mathbb{Q}_3 \times \mathbb{Q}_5$, defined by

$$\varphi_j(\boldsymbol{x}) = C\boldsymbol{x} + \boldsymbol{b}_j,$$

where

$$C = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix},$$

and the translations are

$$\boldsymbol{b}_1 = (0,0), \quad \boldsymbol{b}_2 = (2,0), \quad \boldsymbol{b}_3 = (0,4), \quad \text{and} \quad \boldsymbol{b}_4 = (2,4).$$

These maps are better understood graphically. For details, refer to Figure 6.1. Note that the choices of translations are somewhat arbitrary—the essential requirement in the following discussion is that the Cantor-like set obtained as the attractor of $\{\varphi_1, \varphi_2\}$ is a "vertical" translation of the Cantor-like set obtained as the attractor of $\{\varphi_3, \varphi_4\}$.

For any set $U \subseteq \mathbb{Q}_3 \times \mathbb{Q}_5$, define

$$\Phi(U) = \bigcup_{j=1}^{4} \varphi_j(U),$$

and note that $\mathscr{A}$ is the unique nonempty set such that $\Phi(\mathscr{A}) = \mathscr{A}$. Let Φ^n denote the n-fold composition of Φ with itself, i.e. for any $U \subseteq \mathbb{Q}_3 \times \mathbb{Q}_5$,

$$\Phi^n(U) := \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{n \text{ times}}(U),$$

with the convention that $\Phi^0(U) = U$. For each $n \in \mathbb{N} \cup \{0\}$, let A_n denote the n-th approximant of $\mathscr{A}$, given by

$$A_n := \Phi^n(\mathbb{Z}_3 \times \mathbb{Z}_5).$$

Note that $A_0 = \mathbb{Z}_3 \times \mathbb{Z}_5$. With this notation, the distance zeta function associated to $\mathscr{A}$ is

$$\begin{aligned}
\zeta_{\mathscr{A}}(s) &= \int_{\mathbb{Z}_3 \times \mathbb{Z}_5} d(\boldsymbol{x}, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x}) \\
&= \underbrace{\int_{\mathscr{A}} d(\boldsymbol{x}, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x})}_{= 0, \text{ as } \mu(\mathscr{A}) = 0} + \sum_{n=0}^{\infty} \int_{A_n \setminus A_{n+1}} d(\boldsymbol{x}, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x}) \\
&= \sum_{n=0}^{\infty} \int_{A_n \setminus A_{n+1}} d(\boldsymbol{x}, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x}) \qquad (6.5.1)
\end{aligned}$$

Thus giving an explicit formula for the distance zeta function is reduced to studying the function on the sets $A_n \setminus A_{n+1}$.

Observe that

$$A_n = \Phi^n(\mathbb{Z}_3 \times \mathbb{Z}_5) = \bigcup_{|\boldsymbol{i}|=n} \varphi_{\boldsymbol{i}}(\mathbb{Z}_3 \times \mathbb{Z}_5),$$

where $\varphi_{\boldsymbol{i}}$ is the composition of maps in the IFS according to the word $\boldsymbol{i}$. But

$$\varphi_1^n(\mathbb{Z}_3 \times \mathbb{Z}_5) = 3^n\mathbb{Z}_3 \times 5^n\mathbb{Z}_5,$$

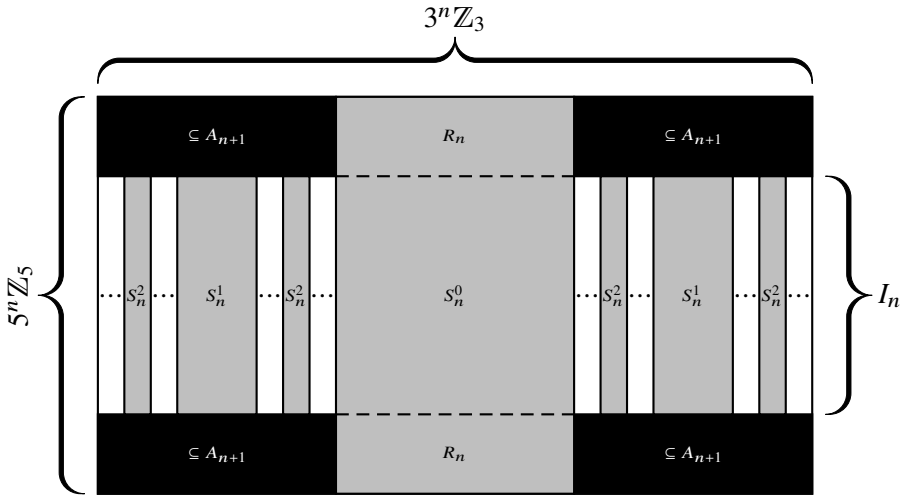

Figure 6.2: One of the 4^{n-1} copies of $3^n\mathbb{Z}_3 \times 5^n\mathbb{Z}_5$ which make up A^{n-1}. The black regions form a subset of A_n, while the light grey regions are sets where the "horizontal" distance to the attractor is greater than the "vertical" distance. Observe that the sets labeled by S_n^k are subsets of the product of a 3-adic Cantor string and the "interval" $I_n \subseteq \mathbb{Z}_5$.

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and if $|\boldsymbol{i}| = n$, then $\varphi_{\boldsymbol{i}}(\mathbb{Z}_3 \times \mathbb{Z}_5)$ is a translation of $\varphi_1^n(\mathbb{Z}_3 \times \mathbb{Z}_5)$. Hence A_n consists of 4^n identical copies of $3^n\mathbb{Z}_3 \times 5^n\mathbb{Z}_5$, and so

$$A_n \setminus A_{n+1}$$

is the set A_n with four copies of $3^{n+1}\mathbb{Z}_3 \times 5^{n+1}\mathbb{Z}_5$ removed from the "corners." As such, $A_n \setminus A_{n+1}$ may be understood as the disjoint union of 4^n copies of $3^n\mathbb{Z}_3 \times 5^n\mathbb{Z}_5$, less the four "corners."

The set $A_n \setminus A_{n+1}$ can be further decomposed into regions where the distance to the attractor is constant (and equal to either a power of 3, or a power of 5). To explicitly describe this decomposition, begin by defining the "interval" I_n as

$$I_n := \bigcup_{j=1}^{3} \left(5^{n+1}\mathbb{Z}_3 + j\right).$$

Let $R_n = (A_n \setminus A_{n+1}) \setminus (3^n\mathbb{Z}_3 \times I_n)$, and note that if $x \in R_n$, then $d(x, \mathscr{A}) = 3^{-n}$. Finally, let

$$S_n^k = \left\{ x \in I_n \times 3^n\mathbb{Z}_3 \;\middle|\; d(x, \mathscr{A}) = 3^{-(n+k)} \right\}.$$

Set (see Fig. 6.2)	Distance to $\mathscr{A}$	No. of Rect.	Vol. of Each Rect.
R_n	3^{-n}	2	$15^{-(n+1)}$
S_n^0	3^{-n}	$2^0 \cdot 3 \cdot 5^0$	$15^{-(n+1)}$
S_n^1	$3^{-(n+1)}$	$2^1 \cdot 3 \cdot 5^1$	$15^{-(n+2)}$
S_n^2	$3^{-(n+2)}$	$2^2 \cdot 3 \cdot 5^2$	$15^{-(n+3)}$
$\vdots$	$\vdots$	$\vdots$	
S_n^k	$3^{-(n+k)}$	$2^k \cdot 3 \cdot 5^k$	$15^{-(n+k+1)}$
$\vdots$	$\vdots$	$\vdots$	
$S_n^{\lfloor n\ell \rfloor}$	$3^{-(n+\lfloor n\ell \rfloor)}$	$2^{\lfloor n\ell \rfloor} \cdot 3 \cdot 5^{\lfloor n\ell \rfloor}$	$15^{-(n+\lfloor n\ell \rfloor+1)}$
$(3^{-n}\mathbb{Z}_3 \times I_n) \setminus \bigcup_k S_n^k$	5^{-n}	$2^{\lfloor n\ell \rfloor+1} \cdot 3 \cdot 5^{\lfloor n\ell \rfloor}$	$15^{-(n+\lfloor n\ell \rfloor+1)}$

Table 6.1: The set $3^n\mathbb{Z}_3 \times 5^n\mathbb{Z}_5 \setminus A_{n+1}$ is decomposed into rectangles in which the distance to the attractor is constant (with respect to the L_∞ metric).

Observe that if k is sufficiently large, then $3^{-(n+k)} < 5^{-n}$. For any such k, the set $S_n^k = \varnothing$. Let k_n denote the greatest integer such that $S_n^{k_n} \neq \varnothing$. Then

$$\begin{aligned} 3^{-(n+k_n+1)} < 5^{-n} < 3^{-(n+k_n)} &\implies \frac{1}{3}\left(\frac{5}{3}\right)^n < 3^{k_n} < \left(\frac{5}{3}\right)^n \\ &\implies n\left(\frac{\log(5)}{\log(3)} - 1\right) - 1 < k_n < \underbrace{n\left(\frac{\log(5)}{\log(3)} - 1\right)}_{=:\ell} \implies k_n = \lfloor n\ell \rfloor. \end{aligned}$$

Finally, if $x \in (3^{-n}\mathbb{Z}_3 \times I_n) \setminus \bigcup_k S_n^k$, then $d(x, \mathscr{A}) = 5^{-n}$. The details of this decomposition are shown in Figure 6.2.

The set S_n^0 consists of three rectangles of the form $[3^{n+1}\mathbb{Z}_3 + 1] \times [5^{n+1}\mathbb{Z}_5 + j]$, where j ranges over $\{1, 2, 3\}$. Each of these rectangles has volume $15^{-(n+1)}$, from which it follows that

$$\begin{aligned} \int_{S_n^0} d(x, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x}) &= \int_{S_n^0} 3^{-n(s-2)} \, \mathrm{d}\mu(\boldsymbol{x}) \\ &= 3^{-n(s-2)} \operatorname{vol}(S_n^0) \\ &= 3^{-n(s-2)}[3 \cdot 15^{-(n+1)}]. \end{aligned}$$

Thus to compute $\int d(x, \mathscr{A})^{s-2}$ over the set shown in Figure 6.2, it is sufficient to determine the volume of each set on which the distance is constant. Since each of these sets can be further decomposed into some number of rectangles each having volume $15^{-(n+k)}$ for some k, a complete description of these decompositions is sufficient. These data are summarized in Table 6.1, leading to an explicit formulation of the distance zeta function, as described in Figure 6.3.

Treat the sums S_1, S_2, and S_3 formally, ignoring issues of convergence. Beginning with S_1,

$$S_1(s) = \sum_{n=0}^{\infty} \left(\frac{4}{3^{s-1} \cdot 5} \right)^n = \frac{3^{s-1} \cdot 5}{3^{s-1} \cdot 5 - 4}, \tag{6.5.2}$$

where equality holds on the open right half-plane on which the series converges absolutely. Hence $S_1(s)$, which *a priori* defines a holomorphic function on the open half-plane

$$\left\{ \Re(s) > \frac{2\log(2) - \log(5)}{\log(3)} + 1 \right\},$$

extends analytically to the mentire function given in (6.5.2). This extended function has poles at

$$s \in \left\{ \frac{2\log(2) - \log(5)}{\log(3)} + 1 + \mathbb{i} \frac{2k\pi}{\log(3)} \;\middle|\; k \in \mathbb{Z} \right\}.$$

To analyze S_2 and S_3, use the Fourier series expansion of $b^{-\{n\ell\}}$, given by

$$b^{-\{n\ell\}} = \sum_{m \in \mathbb{Z}} \alpha_m \exp(2\pi \mathbb{i} m n \ell),$$

where the Fourier coefficients are

$$\begin{aligned}
\alpha_m &= \ell \int_0^{\frac{1}{\ell}} \exp\left(-t(\ell \log(b) + 2\pi \mathbb{i} m \ell)\right) \, \mathrm{d}t \\
&= -\frac{\ell}{\ell \log(b) + 2\pi \mathbb{i} m \ell} \left[\exp(-\log(b) - 2\pi \mathbb{i} m) - 1 \right] \\
&= \frac{1}{\log(b) + 2\pi \mathbb{i} m} \left[1 - \frac{1}{b} \exp(-2\pi \mathbb{i} m) \right] \\
&= \frac{1}{\log(b) + 2\pi \mathbb{i} m} \frac{b-1}{b}. \qquad \text{(since } \mathrm{e}^{-2\pi \mathbb{i} m} = 1 \forall m \in \mathbb{Z}\text{)}
\end{aligned}$$

$$
\begin{aligned}
\zeta_{\mathscr{A}}(s) &= \sum_{n=0}^{\infty} \int_{A_n \setminus A_{n+1}} d(\boldsymbol{x}, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x}) \\
&= \sum_{n=0}^{\infty} \left[4^n \int_{3^n\mathbb{Z}_3 \times 5^n \mathbb{Z}_5 \setminus A_{n+1}} d(\boldsymbol{x}, \mathscr{A})^{s-2} \, \mathrm{d}\mu(\boldsymbol{x}) \right] \\
&= \sum_{n=0} \left[\int_{R_n} 3^{-n(s-2)} \, \mathrm{d}\mu(\boldsymbol{x}) + \left[\sum_{k=0}^{\lfloor n\ell \rfloor} \int_{S_n^k} 3^{-(n+k)(s-2)} \, \mathrm{d}\mu(\boldsymbol{x}) \right] + \int_{(3^{-n} \times I_n) \setminus \bigcup_k S_n^k} 5^{-n(s-2)} \, \mathrm{d}\mu(\boldsymbol{x}) \right] \\
&= \sum_{n=0}^{\infty} 4^n \left[3^{-n(s-2)} \cdot 2 \cdot 15^{-(n+1)} + \sum_{k=0}^{\lfloor n\ell \rfloor} \left[3^{-(n+k)(s-2)} \left[2^k \cdot 3 \cdot 5^k \right] 15^{-(n+k+1)} \right] + 5^{-n(s-2)} \left[2^{\lfloor n\ell \rfloor + 1} \cdot 3 \cdot 5^{\lfloor n\ell \rfloor} \right] 15^{-(n+\lfloor n\ell \rfloor + 1)} \right] \\
&= \sum_{n=0}^{\infty} \left[\frac{2}{15} \left[\frac{4}{3^{s-1} \cdot 5} \right]^n + \frac{1}{5} \left[\frac{4}{3^{s-1} \cdot 5} \right]^n \sum_{k=0}^{\lfloor n\ell \rfloor} \left[\frac{2}{3^{s-1}} \right]^k + \frac{2}{5} \left[\frac{4}{3 \cdot 5^{s-1}} \right]^n \left[\frac{2}{3} \right]^{\lfloor n\ell \rfloor} \right] \\
&= \sum_{n=0}^{\infty} \left[\frac{2}{15} \left[\frac{4}{3^{s-1} \cdot 5} \right]^n + \frac{1}{5} \left[\frac{4}{3^{s-1} \cdot 5} \right]^n \left[\frac{1}{3^{s-1} - 2} \right] \left[3^{s-1} - 2 \left[\frac{2}{3^{s-1}} \right]^{n\ell} \left[\frac{2}{3^{s-1}} \right]^{-\{n\ell\}} \right] + \frac{2}{5} \left[\frac{2^{\ell+2}}{3^{\ell+1} \cdot 5^{s-1}} \right]^n \left[\frac{2}{3} \right]^{-\{n\ell\}} \right] \\
&= \sum_{n=0}^{\infty} \left[\left[\frac{2}{15} + \frac{3^{s-1}}{5(3^{s-1}-2)} \right] \left[\frac{4}{3^{s-1} \cdot 5} \right]^n - \left[\frac{2}{5(3^{s-1}-2)} \right] \left[\frac{2^{\ell+2}}{3^{(\ell+1)(s-1)} \cdot 5} \right]^n \left[\frac{2}{3^{s-1}} \right]^{-\{n\ell\}} + \frac{2}{5} \left[\frac{2^{\ell+2}}{3^{\ell+1} \cdot 5^{s-1}} \right]^n \left[\frac{2}{3} \right]^{-\{n\ell\}} \right] \\
&= \underbrace{\left[\frac{2}{15} + \frac{3^{s-1}}{5(3^{s-1}-2)} \right]}_{=:f_1(s)} \underbrace{\sum_{n=0}^{\infty} \left[\frac{4}{3^{s-1} \cdot 5} \right]^n}_{=:S_1(s)} - \underbrace{\left[\frac{2}{5(3^{s-1}-2)} \right]}_{=:f_2(s)} \underbrace{\sum_{n=0}^{\infty} \left[\frac{2^{\ell+2}}{3^{(\ell+1)(s-1)} \cdot 5} \right]^n \left[\frac{2}{3^{s-1}} \right]^{-\{n\ell\}}}_{=:S_2(s)} + \underbrace{\frac{2}{5} \sum_{n=0}^{\infty} \left[\frac{2^{\ell+2}}{3^{\ell+1} \cdot 5^{s-1}} \right]^n \left[\frac{2}{3} \right]^{-\{n\ell\}}}_{=:S_3(s)}
\end{aligned}
$$

Figure 6.3: The details of the computation of the distance zeta function.

For any $n \in \mathbb{Z}$,

$$b^{-\{n\ell\}} = \frac{b-1}{b} \sum_{m \in \mathbb{Z}} \frac{\exp(2\pi \mathbb{i} mn\ell)}{\log(b) + 2\pi \mathbb{i} m\ell}. \tag{6.5.3}$$

Using (6.5.3) to expand $S_2(s)$ gives

$$\begin{aligned} S_2(s) &= \sum_{n=0}^{\infty} \left(\frac{2^{\ell+2}}{3^{(\ell+1)(s-1)} \cdot 5} \right)^n \left(\frac{2}{3^{s-1}} \right)^{-\{n\ell\}} && (6.5.4) \\ &= \sum_{n=0}^{\infty} \left[\left(\frac{2^{\ell+2}}{3^{(\ell+1)(s-1)} \cdot 5} \right)^n \left(\frac{\frac{2}{3^{s-1}} - 1}{\frac{2}{3^{s-1}}} \right) \sum_{m \in \mathbb{Z}} \frac{\exp(2\pi \mathbb{i} mn\ell)}{\log\left(\frac{2}{3^{s-1}}\right) + 2\pi \mathbb{i} m} \right] \\ &= \frac{2 - 3^{s-1}}{2} \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} \left[\frac{2^{\ell+2} \exp(2\pi \mathbb{i} m\ell)}{3^{(\ell+1)(s-1)} \cdot 5} \right]^n \left[\frac{1}{\log\left(\frac{2}{3^{s-1}}\right) + 2\pi \mathbb{i} m} \right] \\ &\overset{?}{=} \frac{2 - 3^{s-1}}{2} \sum_{m \in \mathbb{Z}} \left[\frac{3^{(\ell+1)(s-1)} \cdot 5}{3^{(\ell+1)(s-1)} \cdot 5 - 2^{\ell+2} \exp(2\pi \mathbb{i} m\ell)} \right] \left[\frac{1}{\log\left(\frac{2}{3^{s-1}}\right) + 2\pi \mathbb{i} m} \right]. && (6.5.5) \end{aligned}$$

The series in (6.5.4) is absolutely convergent—and therefore holomorphic—on the open half-plane

$$\left\{ \Re(s) > \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)} \right\}.$$

The purported identity at (6.5.5) is obtained by exchanging the order of summation and simplifying the resulting geometric series. It is important to note that this exchange cannot be justified by the usual analytic tricks (e.g. the Fubini-Tonelli theorem).

Question 6.9. *Is there any framework under which the exchange of summation* can *be justified? For example, can the series be understood in the setting of distributions or generalized functions* (*in the sense of Schwartz* [Sch66])*? or in the setting of hyperfunctions* (*as outlined by Graf* [Gra10])*?*

An answer to this question is of interest as, assuming that the exchange in the order of summation can be justified, the computation proceeds by observing for each fixed value of m, the summand in (6.5.5) extends to a mentire function with poles at

$$s \in \left\{ \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)} + \mathbb{i} \frac{2\pi(\ell m + k)}{\log(5)} \;\middle|\; k \in \mathbb{Z} \right\} \cup \left\{ \frac{\log(2)}{\log(3)} + 1 - \mathbb{i} \frac{2\pi m}{\log(3)} \right\}.$$

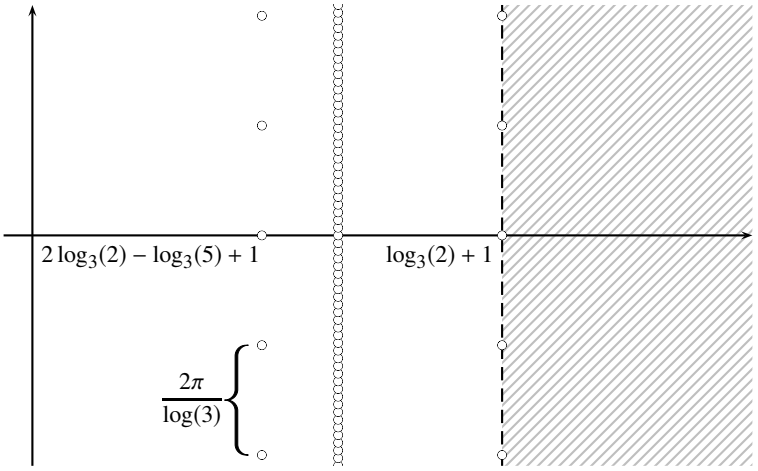

Figure 6.4: The potential poles and other singularities of $\zeta_{\mathscr{A}}(s)$. *A priori* $\zeta_{\mathscr{A}}(s)$ converges absolutely to the right of the abscissa of convergence (dashed). Assuming that the computations can be somehow justified, $\zeta_{\mathscr{A}}$ possesses a dense set of singularities along the line $\Re(s) = \log_3(2) + \log_5(2)$, and cannot be analytically continued to any open domain containing this line.

As there is a term in the series corresponding to each $m \in \mathbb{Z}$, the function $S_2(s)$ has a set of singularities contained in

$$\left\{ \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)} + \mathbb{i}\frac{2\pi(\ell m + k)}{\log(5)} \;\middle|\; k, m \in \mathbb{Z} \right\}. \tag{6.5.6}$$

Since ℓ is irrational, the set $\{2\pi(\ell m + k) \mid k, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Hence if this set describes the singularities of $S_2(s)$, then this series does not extend to a meromorphic function on any domain strictly containing the half-plane of convergence. On the other hand, the potential singularities on the line $\{\Re(s) = \log_3(2) + 1\}$ are exactly canceled by the zeros of the leading term (though $f_2(s)$ does contribute poles, as described below).

The last series can be addressed by a similar (formal) computation, rendering

$$\begin{aligned} S_3(s) &= \sum_{n=0}^{\infty} \left(\frac{2^{\ell+2}}{3^{\ell+1} \cdot 5^{s-1}} \right)^n \left(\frac{2}{3} \right)^{-\{n\ell\}} \\ &= -\frac{1}{2} \sum_{m \in \mathbb{Z}} \left[\frac{3^{\ell+1} \cdot 5^{s-1}}{3^{\ell+1} \cdot 5^{s-1} - 2^{\ell+2} \exp(2\pi \mathbb{i} m \ell)} \right] \left[\frac{1}{\log\left(\frac{2}{3}\right) + 2\pi \mathbb{i} m} \right], \end{aligned} \tag{6.5.7}$$

which again converges absolutely (and is therefore holomorphic) on the open right half-plane

$$\left\{\Re(s) > \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)}\right\},$$

with (potentially) a dense set of singularities (of the form described at (6.5.6)) on the abscissa of convergence.

Finally, note that both $f_1(s)$ and $f_2(s)$ are mentire, with poles at

$$s \in \frac{\log(2)}{\log(3)} + 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}.$$

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