Local Fractal Zeta Functions

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Overview

Geometric Zeta Functions in $\mathbb{R}^n$

Geometric Zeta Functions in $\mathbb{R}^n$: Definitions

Definition

Let $E \subseteq \mathbb{R}^n$ be bounded, and for each $\delta > 0$, let

$$E_\delta := \{x \in \mathbb{R}^n \mid d(x, E) < \delta\}$$

denote a **δ-neighborhood** of E. For any $\delta > 0$, the **distance zeta function** corresponding to E is the complex function

$$\zeta_E(s) := \int_{E_\delta} d(x, E)^{s-n} \, \mathrm{d}x,$$

and the **tube zeta function** corresponding to E is the complex function

$$\tilde{\zeta}_E(s) := \int_0^\delta t^{s-n-1} m(E_t) \, \mathrm{d}t,$$

where $m(E_t)$ denotes the Lebesgue measure of E_t.

Geometric Zeta Functions in $\mathbb{R}^n$: Key properties

- The following functional relation holds whenever both geometric zeta functions are defined:
$$\zeta_E(s) = \delta^{s-n} m(E_\delta) + (n-s)\tilde{\zeta}_E(s).$$
- Both ζ_E and $\tilde{\zeta}_E$ appear to depend on a parameter δ. This dependence is inessential.
- The integrals defining the geometric zeta functions converge absolutely on the open right half-plane
$$\left\{\Re(s) > \overline{\dim}_{\mathrm{Mi}}(E)\right\},$$
where $\overline{\dim}_{\mathrm{Mi}}(E)$ denotes the upper Minkowski dimension of E. These integrals will diverge on the complementary open left half-plane.
- If ζ_E can be meromorphically extended to an open region containing the closure of the half-plane of convergence, the poles of the extension are called the **(visible) complex dimensions** of E. Under relatively mild hypotheses, $\overline{\dim}_{\mathrm{Mi}}(E)$ will be a complex dimension of E.

See Lapidus et. al, *Fractal Zeta Functions and Fractal Drums* [LRŽ17] for further details.

Local Zeta Functions

Local Zeta Functions: The problem

Question

Let (X, d, μ) be a metric measure space with underlying set X, metric d, and Radon measure μ. Is there a natural generalization of geometric zeta functions which can be used to study the geometry of X?

It is tempting to parallel the Euclidean theory, and define a distance zeta function by

$$\zeta_E(s) := \int_{E_\delta} d(x, E)^{s-Q} \, \mathrm{d}\mu(x),$$

where E is a (totally) bounded subset of X, and Q corresponds to some notion of "ambient dimension", i.e. the dimension of X.

Problems:

- The "ambient dimension" is poorly characterized.
- The geometry of a space is described via extrinsic measurements.

Local Zeta Functions: Definitions

An Approach:

"Probe" the space with local zeta functions, then piece the local information together to describe global features.

Definition

Let $X = (X, d, \mu)$ be a complete metric measure space with Radon measure μ. Let $x \in X$ and fix a bounded, open $\Omega \subseteq X$ with $\mu(\Omega) < \infty$ and $x \in \Omega$. The **local distance zeta function** at x relative to Ω is defined as

$$\zeta^{\mathrm{loc}}_{x,\Omega}(s) := \int_\Omega d(y,x)^{-s} \, \mathrm{d}\mu(y).$$

The **local tube zeta function** a x relative to Ω is defined as

$$\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s) := \int_0^{\mathrm{diam}(\Omega)} t^{-s-1} \mu(B(x,t) \cap \Omega) \, \mathrm{d}t.$$

Local Zeta Functions: Key properties

Theorem (H.)

The following functional relation holds whenever both local zeta functions are defined:

$$\zeta^{\mathrm{loc}}_{x,\Omega}(s) = \mathrm{diam}(\Omega)^{-s}\mu(\Omega) + s\tilde{\zeta}^{\mathrm{loc}}_{x,\Omega}(s)$$

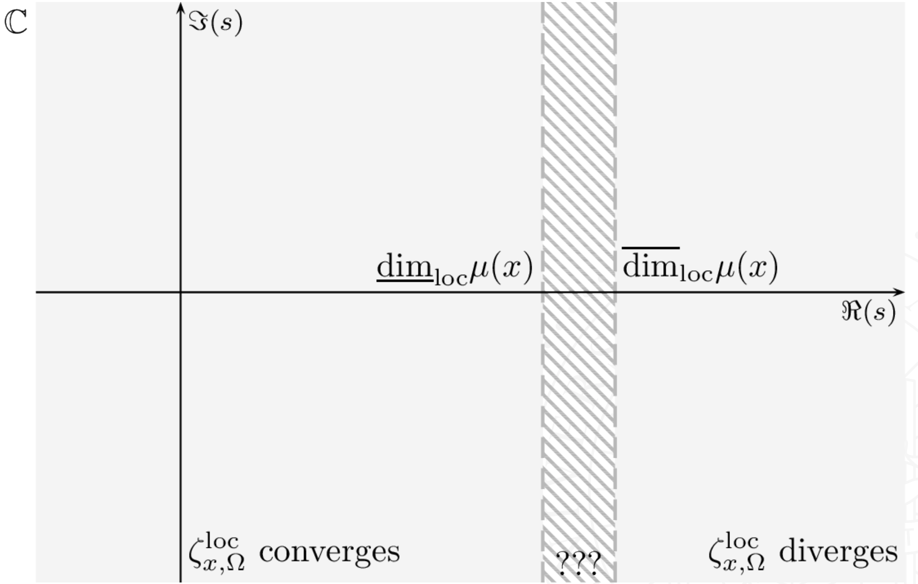
Theorem (H.)

The integrals defining the local zeta functions are (absolutely) convergent on the open left half-plane $\{\Re(s) < \underline{\dim}_{\mathrm{loc}}\mu(x)\}$*, and diverge on the open right half-plane* $\left\{\Re(s) > \overline{\dim}_{\mathrm{loc}}\mu(x)\right\}$*,where*

$$\underline{\dim}_{\mathrm{loc}}\mu(x) := \liminf_{r \searrow 0} \frac{\log(\mu(B(x,r)))}{\log(r)} \quad \textit{and} \quad \overline{\dim}_{\mathrm{loc}}\mu(x) := \limsup_{r \searrow 0} \frac{\log(\mu(B(x,r)))}{\log(r)}$$

denote the lower and upper local dimensions of μ *at* x*. See Falconer* [Fal04] *for a discussion of the local dimension of a measure.*

Local Zeta Functions: Key properties



Local Zeta Functions: Key properties

Definition

Let $X = (X, d, \mu)$ be a complete metric measure space with Radon measure μ, and let $q \geq 0$. The measure μ is said to be *q-homogeneous* if there is a constant $M > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(\xi,\rho))} \leq M \left(\frac{r}{\rho}\right)^q$$

for all $0 < \rho < r \leq 1$, $x \in X$, and $\xi \in B(x,r)$.

Theorem (H.)

Let $x \in X$, suppose that $\dim_{\mathrm{loc}} \mu(x)$ exists, and suppose that there is some neighborhood Ω of x such that the restriction of μ to Ω is D-homogeneous. Then

$$\lim_{\sigma \nearrow D} \zeta^{\mathrm{loc}}_{x,\Omega}(\sigma) = +\infty.$$

Under these hypotheses, the local zeta functions cannot be extended to a function which is analytic on any half-plane larger than $\{\Re(s) < D\}$.

Questions?

References

[Fal04] Kenneth Falconer.
Fractal Geometry: Mathematical Foundations and Applications.
Wiley, 2004.

[LRŽ17] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić.
Fractal Zeta Functions and Fractal Drums.
Springer, 2017.