Homogeneous Spaces and the Assouad Dimension

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The following notes are taken (almost word-for-word, in many places) from:

James C. Robinson, *Dimensions, embeddings, and attractors*, Cambridge University Press, Cambridge, 2011.

1 Homogeneity

Definition 1.1. Let (X, d) be a metric space and $A \subseteq X$. Given positive constants M and s, we say that A is *(M,s)-homogeneous* if the intersection of A with any r-ball can be covered by $M(r/\rho)^s$ or fewer ρ-balls, where $\rho < r$. More generally, we will say that a set is *homogeneous* if it is (M, s)-homogeneous for some M and s.

For notational convenience, we define notation for "ball counting." Given $A \subseteq X$, let $\mathcal{N}(A; r)$ denote the number of r-balls required to cover A, and (informally) let $\mathcal{N}(A; r, \rho)$ denote the maximal number of ρ-balls required to cover an r-ball centered in A. More exactly,

$$\mathcal{N}(A; r, \rho) := \sup_{x \in A} \mathcal{N}(A \cap B(x, r); \rho),$$

The condition of homogeneity can then be expressed as follows: a set $A \subseteq X$ is (M, s)-homogeneous if

$$\mathcal{N}(A; r, \rho) \leq M \left(\frac{r}{\rho} \right)^s \tag{1.1}$$

for all $0 < \rho < r$.

Example 1.2. *Every subset of* $\mathbb{R}^N$ *is* $((4\sqrt{N})^N, N)$*-homogeneous.*

Proof. Consider the ball of radius r centered at the origin. This ball is contained in the cube $[-r, r]^N$. This cube can be covered by $[(2r\sqrt{N}/\rho) + 1]^N =: K$ cubes of side length $\rho/\sqrt{N}$, where $\rho < r$—let $\{C_i\}$ denote this collection of cubes. Each cube C_i is contained in some ρ-ball B_i, hence

$$B(0, r) \subseteq [-r, r]^N \subseteq \bigcup_{i=1}^{K} C_i \subseteq \bigcup_{i=1}^{K} B_i.$$

Then, in the notation developed above, $\mathcal{N}(B(0,r);\rho) \leq K$, hence we seek to bound K. It is an exercise to show that

$$K = \left(\frac{2r}{\rho/\sqrt{N}} + 1\right)^N \leq (4\sqrt{N})^N \left(\frac{r}{\rho}\right)^N,$$

from which it follows that $\mathcal{N}(B(0,r);\rho) \leq (4\sqrt{N})^N (r/\rho)^N$. Translating an r-ball away from the origin will not change any of the above analysis, and the intersection of an r-ball with a set will not require more ρ-balls to cover, thus for any $A \subseteq \mathbb{R}^N$, any $x \in A$, and any $0 < \rho < r$, we have

$$\mathcal{N}(A;r,\rho) \leq \mathcal{N}(A \cap B(x,r);\rho) \leq \mathcal{N}(B(x,r);\rho) \leq (4\sqrt{N})^N \left(\frac{r}{\rho}\right)^N.$$

That is, as per the condition given at (1.1), any subset of $\mathbb{R}^N$ is $((4\sqrt{N})^N, N)$-homogeneous. □

We note that in the above, $(4\sqrt{N})^N$ is not sharp. Robinson claims that the result holds if we replace $(4\sqrt{N})^N$ with 2^{N+1}. However, as discussed below, the scaling constant is inessential for our purposes, hence we aren't terribly concerned with obtaining sharp bounds.

Proposition 1.3. *Suppose that (X, d_X) and (Y, d_Y) are metric spaces, that X is (M,s)-homogeneous, and that the map $f : X \to Y$ is bi-Lipschitz, i.e. there is some $L > 0$ such that*

$$L^{-1} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. Then $f(X)$ is a (ML^{2s}, s)-homogeneous subset of Y.

Proof. Let $y \in f(X)$ and consider $f(X) \cap B(y,r)$. Let $x = f^{-1}(y)$. As f is bi-Lipschitz, it is invertible and, moreover, we have

$$f^{-1}(f(X) \cap B(y,r)) \subseteq B(x, Lr).$$

As X is (M,s)-homogeneous, for any $\rho < r$ we have

$$\mathcal{N}\Big(B(x,Lr); \frac{\rho}{L}\Big) \leq M\left(\frac{Lr}{\rho/L}\right)^s = ML^{2s}\left(\frac{r}{\rho}\right)^s =: K.$$

That is, there is a collection of at most K balls of the form $B(x_j, \rho/L)$ such that

$$B(x, Lr) \subseteq \bigcup_{j=1}^{K} B(x_j, \rho/L).$$

Mapping forward with f, we obtain

$$f(X) \cap B(y,r) \subseteq f(B(x,Lr)) \subseteq f\left(\bigcup_{j=1}^{K} B(x_j, \rho/L)\right) \subseteq \bigcup_{j=1}^{K} B(f(x_j), \rho).$$

As we have covered the intersection of $f(X)$ and an arbitrary r-ball with K (or fewer) ρ-balls, and the choices of r and ρ were arbitrary, we have have

$$\mathcal{N}(f(X);\rho) \leq K = ML^{2s}\left(\frac{r}{\rho}\right)^s.$$

Again, as per condition (1.1), we have that $f(X)$ is (ML^{2s}, s)-homogeneous. □

The punchline here is that homogeneity is preserved under bi-Lipschitz mappings. Moreover, in light of example 1.2, this is sufficient to show that homogeneity is a necessary (though, as we'll discuss later, not sufficient) condition for the existence of a bi-Lipschitz embedding.

Definition 1.4. A set $A \subseteq (X, d)$ is *doubling* if there exists some $C > 0$ such that

$$\mathcal{N}(A; r, r/2) \leq C$$

for all $r > 0$.

Proposition 1.5. *A set $A \subseteq (X, d)$ is homogeneous if and only if it is doubling.*

Proof. First, suppose that A is (M, s)-homogeneous. Then

$$\mathcal{N}(A; r, r/2) \leq M\left(\frac{r}{r/2}\right)^s = 2^s M,$$

and so A is doubling with constant $C = 2^s M$.

Conversely, suppose that A is doubling, so that $\mathcal{N}(A; r, r/2) \leq C$ for all $r > 0$. Fix some $\rho < r$ and choose n such that

$$\frac{r}{2^n} \leq \rho < \frac{r}{2^{n-1}}.$$

Note that this implies

$$\log_2\left(\frac{r}{\rho}\right) > n - 1. \tag{1.2}$$

Given an arbitrary $x \in A$, we may cover $A \cap B(x, r)$ with ρ-balls by first covering it with $r/2$-balls, then covering each $r/2$-ball with $r/4$-balls, and so on. This gives the following computation:

$$\begin{aligned}
\mathcal{N}(A; r, \rho) &= \mathcal{N}(A; r, r/2) \cdots \mathcal{N}(A; r/2^{n-2}, r/2^{n-1})\, \mathcal{N}(A; r/2^{n-1}, \rho) \\
&\leq \mathcal{N}(A; r, r/2) \cdots \mathcal{N}(A; r/2^{n-2}, r/2^{n-1})\, \underbrace{\mathcal{N}(A; r/2^{n-1}, r/2^n)}_{\geq \mathcal{N}(A; r/2^{n-1}, \rho)} \\
&\leq C^n \\
&= CC^{n-1} \\
&\leq CC^{\log_2(r/\rho)} && \text{(as per the estimate at (1.2))} \\
&= C\left(\frac{r}{\rho}\right)^{\log_2(C)}.
\end{aligned}$$

Thus A is $(C, \log_2(C))$-homogeneous. □

2 Assouad Dimension

In most applications, the scaling constant M plays very little role, hence it is natural to make the following definition:

Definition 2.1. The *Assouad dimension* of a space (X, d), denoted $\dim_A(X)$, is the infimal s such that (X, d) is (M, s)-homogeneous for some $M \geq 1$.

The following proposition lists several basic properties of the Assouad dimension. Note that (a) and (b) follow very quickly from the definition, while (c) was proved in proposition 1.3.

Proposition 2.2.

(a) If $A \subseteq B \subseteq (X, d)$*, then* $\dim_A(A) \leq \dim_A(B)$.

(b) If $A, B \subseteq (X, d)$*, then* $\dim_A(A \cup B) \leq \max(\dim_A(A), \dim_A(B))$.

(c) $\dim_A$ *is invariant under bi-Lipschitz mappings.*

(d) If $X \subseteq \mathbb{R}^N$ *is open, then* $\dim_A(X) = N$.

(e) If X *is compact, then* $\dim_{UB}(X) \leq \dim_A(X)$*, where* $\dim_{UB}$ *denote the upper box-counting dimension.*

Proof (d). It was shown in example 1.2 that $\mathbb{R}^N$ is homogeneous with exponent N, hence $\dim_A(\mathbb{R}^N) \leq N$. As the Assouad dimension is monotone (part (a) of the current proposition), it follows that $\dim_A(X) \leq N$. Let $B \subseteq X$ be an open ball with radius r, and suppose for contradiction that $\dim_A(B) < s \leq \dim_A(X) < N$. But then B is (M, s)-homogeneous for some $M \geq 1$. But then B can be covered by $M(r/\rho)^s$ balls of radius ρ, hence

$$\mu(B) \leq M\left(\frac{r}{\rho}\right)^s \mu(B(0, \rho)) \leq M\Omega_N \left(\frac{r}{\rho}\right)^s \rho^N = M\Omega_N r^s \rho^{N-s}.$$

But $N - s > 1$, and $\rho > 0$ is arbitrary, which implies that $\mu(B) = 0$. This is a contradiction, hence we have $N \leq \dim_A(B) \leq \dim_A(X) \leq N$. □

Proof (e). Recall that

$$\dim_{UB}(X) \leq \limsup_{\varepsilon \to 0} \frac{\log(\mathcal{N}(X; \varepsilon))}{\log(1/\varepsilon)},$$

and let $s > \dim_A(X)$, from which it follows that X is (M, s)-homogeneous for some $M \geq 1$. As X is compact, it is bounded, and so there is some $R > 0$ such that $X \subseteq B(0, R)$. It then follows that for any $\rho < R$, we have

$$\mathcal{N}(X; \rho) = \mathcal{N}(X \cap B(0, R); \rho) \leq M\left(\frac{R}{\rho}\right)^s = (MR^s)\rho^{-s}.$$

But then

$$\limsup_{\varepsilon \to 0} \frac{\log(\mathcal{N}(X; \varepsilon))}{\log(1/\varepsilon)} = \lim_{\rho \to 0} \frac{\log(\mathcal{N}(X; \rho))}{\log(1/\rho)} \leq \lim_{\rho \to 0} \frac{\log[(MR^s)\rho^{-s}]}{\log(1/\rho)} = s,$$

which gives the desired result. □

Proposition 2.3. *If (X, d_X) and (Y, d_Y) are metric spaces, then*

$$\dim_A(X \times Y) \leq \dim_A(X) + \dim_A(Y),$$

where $X \times Y$ is equipped with any metric d_α of the form

$$d_\alpha((x_1, y_1), (x_2, y_2)) = [d_X(x_1, x_2)^\alpha + d_Y(y_1, y_2)^\alpha]^{1/\alpha}$$

for some $\alpha \in [1, \infty)$, or the metric

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

Proof. For any $\alpha, \beta \in [1, \infty]$, the metrics d_α and d_β are equivalent, and so the space $(X \times Y, d_\alpha)$ can be mapped to the space $(X \times Y, d_\beta)$ via a bi-Lipschitz map. As the Assouad dimension is invariant under such maps, we may assumed without loss of generality that $X \times Y$ is equipped with the d_∞ metric.

Assume that $s > \dim_A(X)$ and $t > \dim_A(Y)$. Then there are constants $M, N \geq 1$ such that

$$\mathcal{N}(X; r, \rho) \leq M \left(\frac{r}{\rho}\right)^s, \quad \text{and} \quad \mathcal{N}(Y; r, \rho) \leq N \left(\frac{r}{\rho}\right)^t.$$

Let B be a ball of radius r in $X \times Y$. Then, as we have assumed that $X \times Y$ is equipped with the d_∞ metric, it follows that $B = U \times V$, where U and V are balls of radius r in X and Y, respectively.

We may cover U by a collection $\{U_i\}$ of at most $\mathcal{N}(X; r, \rho)$ balls of radius ρ, and we may cover V by a collection $\{V_j\}$ of at most $\mathcal{N}(Y; r, \rho)$ balls of radius ρ. But then the collection $\{U_i \times V_j\}$ is a cover of B which contains at most

$$\mathcal{N}(X; r, \rho)\mathcal{N}(Y; r, \rho) \leq M \left(\frac{r}{\rho}\right)^s N \left(\frac{r}{\rho}\right)^t = MN \left(\frac{r}{\rho}\right)^{s+t}$$

balls of radius ρ. Hence $X \times Y$ is $(MN, s+t)$-homogeneous, which completes the proof. □

Unlike the Hausdorff and upper box-counting dimensions, the Assouad dimension can be quite poorly behaved with respect to orthogonal sequences. The following proposition gives an example of a sequence with infinite Assouad dimension:

Proposition 2.4. *Let $\{e_n\}$ be an orthonormal sequence in a Hilbert space, and let $X = \{n^{-\alpha} e_n : n \in \mathbb{N}\} \cup \{0\}$, where $\alpha > 0$. Then $\dim_A(X) = \infty$.*

Proof. For each $m \in \mathbb{N}$, let $r_m = m^{-\alpha}$ and consider the set

$$X \cap B(0, r_m) = \{n^{-\alpha} e_n : n \geq m\} \cup \{0\}.$$

We seek to cover this set by balls of radius $r_m/2$. Every point in this set that has norm greater than $r_m/2$ will require a separate ball, so we estimate the number of such balls. That is, we need to know how many n satisfy the inequality

$$\frac{r_m}{2} < \|n^{-\alpha} e_n\| \leq r_m.$$

Equivalently,

$$\frac{m^{-\alpha}}{2} < n^{-\alpha} \leq m^{-\alpha} \Longrightarrow 2^{1/\alpha}m > n \geq m.$$

There are at least $2^{1/\alpha}m - 1 - m$ integers n that satisfy this inequality, hence

$$\mathcal{N}(X; r_m, r_m/2) \geq 2^{1/\alpha}m - 1 - m = m(2^{1/\alpha} - 1) - 1.$$

As the quantity on the right-hand side tends to infinity as $m \to \infty$, it follows that X cannot be a doubling set. Hence by proposition 1.5, X is not (M, s)-homogeneous for any s. Therefore $\dim_A(X) = \infty$, as claimed. □

In contrast to this result, it can be shown that $\dim_H(X) = 0$ ($\dim_X(X)$ denotes the Hausdorff dimension of X) and that $\dim_{UB}(X) = 1/\alpha$. Several other results involving orthogonal sequences are of interest.

Proposition 2.5. *Let $\{e_n\}$ be an orthonormal sequence in a Hilbert space, and consider the set $X = \{a_n e_n\}_{n=1}^{\infty} \cup \{0\}$.*

(a) If there is some $K > 0$ and $0 < \alpha < 1$ such that $K^{-1}\alpha^n \leq a_n \leq K\alpha^n$, then $\dim_A(X) = 0$.

(b) There exist sequences $\{a_n\}$ converging to zero arbitrarily quickly such that $\dim_A(X) = \infty$, hence the lower bound in part (a) is necessary.

Proposition 2.6. *There exists a subset X of Hilbert space with $\dim_A(X) = 0$ and $\dim_A(X - X) = \infty$, where $X - X = \{x - y : x, y \in X\}$.*

This result is of interest for at least two reasons. First, it demonstrates that the Assouad dimension can increase under Lipschitz maps since $\dim_A(X \times X) \leq 2\dim_A(X)$, and $X - X$ is the image of $X \times X$ under the Lipschitz mapping $(x, y) \mapsto x - y$. Hence the lower bound in proposition 1.3 really is necessary.

Second, many embedding results that use linear maps rest on strong dimensional assumptions on the set of differences $X - X$. The upper box-counting dimension is well-behaved in the sense that $\dim_{UB}(X - X) \leq 2\dim_{UB}(X)$. Unfortunately, the Assouad dimension, like the Hausdorff dimension, displays the kind of "pathology" described above. Finally:

Proposition 2.7. *Let $X = \{x_n\}_{n=1}^{\infty}$ be an orthogonal sequence in Hilbert space. If $\dim_A(X) < \infty$, then $\dim_A(X - X) \leq 2\dim_A(X)$.*

3 The Laakso Graph

It was noted above that if (X, d) is a metric space, then the homogeneity of X is a necessary condition for the existence of a bi-Lipschitz embedding of X into Euclidean space. The Laakso graph gives an example of a homogeneous metric space that cannot be embedded, demonstrating that homogeneity is not a sufficient condition.

To construct the Laakso graph, first let $\Gamma_0 = [0, 1]$. To obtain Γ_{j+1}, take six copies of Γ_j and scale them by $1/4$. Identify endpoints of four of these copies

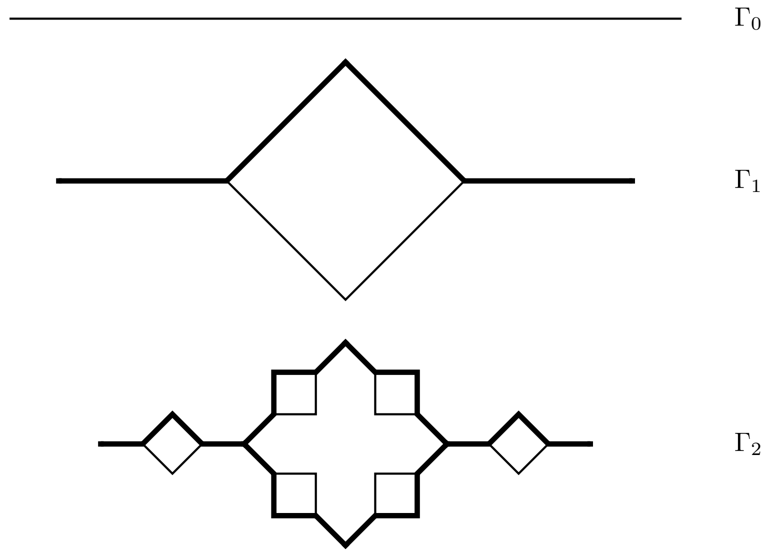

Figure 3.1: The first three steps in the construction of the Laakso graph. The heavier lines in Γ_1 show an isometric copy of Γ_0, while the heavier lines in Γ_2 show an isometric copy of Γ_1.

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to form a "square," and attach the remaining two copies to opposite vertices of this square. The first three steps are shown in figure 3.1.

At the j-th step, the graph consists of 6^j edges of length 4^{-j}. A natural metric d_j on Γ_j is given by geodesic distance, i.e. if $x, y \in \Gamma_j$ then $d_j(x, y)$ is the minimal length of a path from x to y. Note that Γ_j is isometrically embedded into Γ_{j+1} for each j. Again, see figure 3.1. The sequence of metric spaces $\{(\Gamma_j, d_j)\}_{j=0}^{\infty}$ is Cauchy, and therefore converges, in the Gromov-Hausdorff metric. Let (Γ, d) denote this limit space. Note that (Γ, d) contains isometric copies of (Γ_j, d_j) for each j.

We claim without proof that (Γ, d) is a doubling space with constant 6. In particular, this implies that (Γ, d) is homogeneous. However, as we will show below, (Γ, d) cannot be embedded into any finite dimensional Euclidean space via a bi-Lipschitz map. We first require the following lemma:

Lemma 3.1. *Let $\mathscr{H}$ denote Hilbert space, and suppose that $f : \Gamma \to \mathscr{H}$ satisfies the property that $\|f_j(x) - f_j(y)\| \geq d_j(x, y)$ for each j, where f_j is any restriction of f to an isometric copy of Γ_j in Γ. Then there exists a pair of consecutive vertices $x, x' \in \Gamma_j$ such that*

$$\|f(x) - f(x')\| \geq \left(1 + \frac{j}{4}\right) d_j(x, x')^2. \tag{3.1}$$

Proof. The proof is by induction. The inequality at (3.1) holds by hypothesis when $j = 0$, establishing a base for induction. Now suppose that there is some $j-1 > 0$ such that (3.1) holds for some consecutive pair of vertices $x, x' \in \Gamma_{j-1}$. As Γ_j contains an isometric copy of Γ_{j-1}, the vertices x and x' correspond to some points x_0 and x_2 in Γ_j, respectively, so that

$$\|f(x_0)-f(x_2)\| \geq \left(1+\frac{j-1}{4}\right) d_{j-1}(x_0,x_2)^2 = \left(1+\frac{j-1}{4}\right) d_j(x_0,x_2)^2. \quad (3.2)$$

Let x_1 and x_3 be the two midpoints between x_0 and x_2. Setting $x_4 := x_0$, we then obtain

$$\begin{aligned}\sum_{k=1}^{3} \|f(x_k)-f(x_{k+1})\|^2 &\geq \|f(x_0)-f(x_2)\|^2 + \|f(x_1)-f(x_3)\|^2 \\ &\qquad \text{(“quadrilateral inequality”)} \\ &\geq \left(1+\frac{j-1}{4}\right) d_j(x_0,x_2)^2 + d_j(x_1,x_3)^2 \\ &\qquad \text{(by (3.2); hypothesis on } f) \\ &= 4\left(1+\frac{j}{4}\right) d_j(x_1,x_3)^2. \\ &\qquad \text{(since } 2d_j(x_1,x_3) = d_j(x_0,x_2))\end{aligned}$$

It then follows from the pigeonhole principle that there is some $k \in \{0,1,2,3\}$ such that

$$\|f(x_k)-f(x_{k+1})\|^2 \geq \left(1+\frac{j}{4}\right) d_j(x_1,x_3)^2.$$

Take x' to be the midpoint between x_k and x_{k+1}. It follows from the triangle inequality that

$$\begin{aligned}\|f(x_k)-f(x')\|^2 + \|f(x')-f(x_{k+1})\|^2 &\geq \left(1+\frac{j}{4}\right) d_j(x_k,x_{k+1}) \\ &= \left(1+\frac{j}{4}\right) \left[d_j(x_k,x') + d_j(x',x_{k+1})\right].\end{aligned}$$

By another application of the pigeonhole principle, we may take x to be either x_k or x_{k+1} in order to obtain (3.1). □

Proposition 3.2. *The metric space* (Γ, d) *cannot be embedded into finite dimensional Euclidean space via a bi-Lipschitz map.*

Proof. Let $f : \Gamma \to \mathscr{H}$, and suppose that there is some $L > 0$ such that

$$L^{-1} d(x,y) \leq \|f(x)-f(y)\| \leq L d(x,y).$$

But then $g : \Gamma \to \mathscr{H}$ defined by $g(x) = Lf(x)$ is a map that, when restricted to any isometric copy of Γ_j, satisfies $d(x,y) \leq \|g(x)-g(y)\|$. By lemma 3.1,

$$\left(1+\frac{j}{4}\right)^{1/2} d(x,y) \leq \|g(x)-g(y)\| \leq L^2 d(x,y) \qquad \forall j,$$

which is a contradiction. □