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Assouad Dimension and the Open Set Condition

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by

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Abstract

The Assouad dimension is a measure of the complexity of a fractal set similar to the box counting dimension, but with an additional scaling requirement. In this thesis, we generalize Moran's open set condition and introduce a notion called grid like which allows us to compute upper bounds and exact values for the Assouad dimension of certain fractal sets that arise as the attractors of self-similar iterated function systems. Then for an arbitrary fractal set $\mathcal{A}$, we explore the question of whether the Assouad dimension of the set of differences $\mathcal{A} - \mathcal{A}$ obeys any bound related to the Assouad dimension of $\mathcal{A}$. This question is of interest, as infinite dimensional dynamical systems with attractors possessing sets of differences of finite Assouad dimension allow embeddings into finite dimensional spaces without losing the original dynamics. We find that even in very simple, natural examples, such a bound does not generally hold. This result demonstrates how a natural phenomenon with a simple underlying structure has the potential to be difficult to measure.

To Wendryn and Yekaterina.

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Chapter 1

Introduction

The work presented in this thesis was originally motivated by two questions: Does the Moran open set condition provide sufficient structure for the Assouad dimension of a self-similar fractal set to be computed? and can bounds be found on the Assouad dimension of $\mathcal{A} - \mathcal{A}$ in terms of the Assouad dimension of $\mathcal{A}$ in the case that $\mathcal{A}$ is a fractal set with additional structure and regularity? The first question was originally posed in 2008 by Luukkainen [15], and was answered by Mackay in 2010 [18] using the theory of Ahlfors regular measures. The second question is motivated by the projection theorem of Olson and Robinson [23]

In Chapter 3, we use elementary techniques to answer to the first question, while simultaneously computing bounds on the Assouad dimension for the class of fractal sets that have not been considered by other authors, namely those sets that occur as the attractors of grid like iterated function systems. In particular, we introduce the notion of a grid like iterated function system, and prove the Assouad dimension of the attractor of such a system is bounded above by an analog of the similarity dimension. In the case that the system has a self-similar attractor and satisfies the more restrictive Moran open set condition, the bound is sharp, and the Assound dimension is equal to the similarity dimension.

To gain some intuition, we give several examples in Chapter 4. Through these examples, we demonstrate the generality of the notion of grid like systems, and use this notion to compute bounds on, and in some cases exact values of, the Assouad dimension of certain self-similar sets.

Finally in Chapter 5, we apply the results of Chapters 3 and 4 to address a question of practical interest. Provided that the attractor of a dynamical system in Hilbert space has a set of differences with finite Assouad dimension, the attractor can be embedded into finite dimensional Euclidean space without losing information about the original dynamics. It is known that there are examples of sets of arbitrarily small Assouad dimension with sets of difference of infinite Assouad dimension. It might be hoped that if a fractal set has enough structure and regularity, its set of differences might have an Assouad dimension which obeys some bound in terms of the Assouad dimension of the original set. We show that simple fractal sets which occur naturally as the result of highly structured iterated function systems can be constructed so that no such bound exists.

Chapter 2

Background

2.1 Fractals and Fractal Dimension

In his seminal *The Fractal Geometry of Nature* [19, page 15], Mandelbrot defines a fractal to be a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension. In the the second printing, he added that it would be best to leave the term "fractal" without a pedantic definition, to use "fractal dimension" as a generic term, and to use in each specific case whichever definition is the most appropriate [19, page 459].

This sentiment is echoed by Luukkainen [16], who states that a *fractal* might be defined as a non-empty compact metric space X for which at least one of the inequalities

$$\dim X \leq \dim_H(X) \leq \underline{\dim}_B(X) \leq \dim_f(X) \leq \dim_A(X)$$

for the topological, Hausdorff, lower box-counting, fractal (upper box-counting), and Assouad dimension, respectively, of X is strict. In contrast, if none of the above inequalities are strict, Luukkainen terms X an *antifractal*. He then proves that for any compact metric space X, a metric always exists such that X is an antifractal with respect to that metric [16, Theorem 4.3]. Thus being a fractal is not a set theoretic or topological property of a space, but is instead a property of a metric space.

As noted by Mandelbrot and Luukkainen, there are many notions of dimension that can be used to quantify the complexity of a set in terms of its metric and topological properties. In this thesis, we are primarily interested in the fractal or upper box-counting dimension (see, for instance, [8][24]), as well as the Assouad or Bouligand dimension (see, for instance, [1][3]).

2.2 Iterated Function Systems

Where possible, we adopt notation and definitions similar to those outlined by Falconer [7] and [8] in our discussion of iterated function systems and their attractors. These are briefly presented below.

Let $f\colon \mathbb{R}^D \to \mathbb{R}^D$ be a continuous map. The function f is said to be a *contraction* if there is some $c \in (0,1)$ such that $|f(x) - f(y)| \leq c|x-y|$ for all $x, y \in \mathbb{R}^D$. If equality holds and $|f(x) - f(y)| = c|x-y|$ for all $x, y \in \mathbb{R}^D$, then f is called a *contracting similarity*, or more concisely, a *similarity*. A similarity maps sets in $\mathbb{R}^D$ to geometrically similar sets by f. In either case, c is called the *contraction ratio* of f.

Let $F = \{f_i\}_{i=1}^{L}$ be a finite collection of continuous maps on $\mathbb{R}^D$. If $L \geq 2$ and f_i is a contraction with contraction ratio c_i for each $i = 1, 2, \ldots, L$, we shall call F an *iterated function system*. It is useful to be able to discuss the image of a set $X \subseteq \mathbb{R}^D$ under sequences of contractions from F. In the simplest case, for any $p \in \mathbb{N}$, we take

$f_i^p(X)$ to mean the image of X under the p-fold composition of f_i. That is,

$$f_i^p(X) = \underbrace{f_i \circ f_i \circ \cdots \circ f_i}_{p\text{-times}}(X).$$

For more intricate compositions, additional notation is required. Let $\mathcal{S}_L$ be the set of all finite sequences of integers between 1 and L, inclusive. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{S}_L$, then $\ell(\alpha) = m$ is the length of α, the sequence of compositions determined by α is

$$f^\alpha(X) = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_m}(X),$$

and the contraction ratio of f^α is

$$c_\alpha = \prod_{i=1}^{m} c_{\alpha_i}.$$

Definition 2.2.1. For convenience, we denote by ϕ the *empty sequence*, where $\ell(\phi) = 0$. While ϕ is of finite length, we do not consider ϕ to be an element of $\mathcal{S}_L$. The set $f_\phi(X)$ is the image of X under no mappings, hence f_ϕ is the identity function. For similar reasons, it is clear that $c_\phi = 1$. It is sometimes useful to truncate a sequence by a single term. Hence throughout this thesis, we adopt the notation $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$. Note that ϕ' is undefined.

Falconer [8, Theorem 9.1] shows that each iterated function system determines a unique, non-empty, compact set $\mathcal{A} \subseteq \mathbb{R}^D$ such that

$$\mathcal{A} = \bigcup_{i=1}^{L} f_i(\mathcal{A}). \tag{2.2.1}$$

$\mathcal{A}$ is referred to as the *invariant set* (or *attractor*) of F. If F is an iterated function system of contracting similarities, then the invariant set $\mathcal{A}$ is called *self-similar*, and the *similarity dimension* of $\mathcal{A}$, $\dim_s(\mathcal{A})$, is the number s such that $\sum_{i=1}^{L} c_i^s = 1$.

Of practical interest is a quantification of the complexity of the attractor of an iterated function system. The fractal dimension gives such a measurement.

Definition 2.2.2. Let (X, d) be a metric space, and $\mathcal{A} \subseteq X$. Let $N(\mathcal{A}, \varepsilon)$ denote the minimum number of closed balls of radius ε with centers in $\mathcal{A}$ required to cover $\mathcal{A}$. The *upper box-counting dimension* or *fractal dimension* of $\mathcal{A}$ is

$$\dim_f(\mathcal{A}) = \limsup_{\varepsilon \to 0} \frac{\log(N(\mathcal{A}, \varepsilon))}{-\log \varepsilon}. \tag{2.2.2}$$

Note the subscripted f, which we use to distinguish this notion of fractal dimension from other dimensions that we discuss. A useful tool for computing fractal dimension is the Moran open set condition.

Definition 2.2.3. Let F be an iterated function system, and suppose that there is a nonempty, bounded, open set U such that

(1) $f_i(U) \cap f_j(U) = \emptyset$ for $i \neq j$, and

(2) $f_i(U) \subseteq U$ for all $i = 1, 2, \ldots, L$.

Then F is said to satisfy the *Moran open set condition*, or simply the *open set condition*. The set U is termed a *Moran open set* (see [6, page 191]). Note that U need not be unique, and that for any given iterated function system, there may be many Moran open sets.

This condition ensures that when the attractor is mapped by the functions which comprise F, the images do not overlap too much. Moreover, as shown in Falconer [7, page 122], $\mathcal{A} \subseteq \overline{U}$, from which it follows that $\operatorname{diam}(\mathcal{A}) \leq \operatorname{diam}(\overline{U})$. Note that there are examples of both equality and strict inequality.

Of significance is the following result: if $\mathcal{A}$ is the self-similar attractor of an iterated function system of which satisfies the open set condition, then the similarity dimension and the fractal dimension of $\mathcal{A}$ coincide. That is,

$$\dim_f(\mathcal{A}) = \dim_s(\mathcal{A}), \tag{2.2.3}$$

where $\dim_f(\mathcal{A})$ is the fractal dimension, as defined in Definition 2.2.2. For discussion, see [7, Theorem 9.3] or [8, Theorem 8.6].

2.3 Assouad Dimension

A notion of dimension similar to the fractal dimension with an additional geometric scaling was defined by Assouad [1] as follows:

Definition 2.3.1. Given a subset $\mathcal{A}$ of a metric space (X, d), the *Assouad dimension of* $\mathcal{A}$, denoted $\dim_A(\mathcal{A})$, is defined as

$$\dim_A(\mathcal{A}) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \Delta_{\varepsilon,t}(\mathcal{A}),$$

where

$$\Delta_{\varepsilon,t}(\mathcal{A}) = \sup \left\{ \frac{\log \mathcal{N}_{\mathcal{A}}(r, \rho)}{\log(r/\rho)} \,\middle|\, 0 < \rho < r < \varepsilon \text{ and } r > t\rho \right\}$$

and $\mathcal{N}_{\mathcal{A}}(r, \rho)$ is the number of balls of radius ρ required to cover any ball of radius r in $\mathcal{A}$ (see also Movahedi-Lankarani [21, Definition 3.1] or Olson [22, Definition 2.2]).

As noted by Lankarani [21], the Assouad dimension behaves as we would expect with respect to subsets: if $\mathcal{A} \subseteq \mathcal{B}$, then $\dim_A(\mathcal{A}) \leq \dim_A(\mathcal{B})$. This proves to be a useful tool for computing lower bounds for the Assouad dimension of a set.

Luukkainen [16, Theorem A.5(11)] shows that for any compact set $\mathcal{A}$, a lower bound for the Assouad dimension is given by

$$\dim_f(\mathcal{A}) \leq \dim_A(\mathcal{A}). \tag{2.3.1}$$

In order to compute upper bounds for the Assouad dimension we take advantage of the following characterization, which appears in Olson [22, Theorem 2.3]: The Assouad dimension of a compact set $\mathcal{A}$ is the infimal value of a for which there exists a constant K such that for any r and ρ with $0 < \rho < r < 1$,

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \leq K \left(\frac{r}{\rho}\right)^a. \tag{2.3.2}$$

This characterization of the Assouad dimension relates to the notion of homogeneity. If a set $\mathcal{A}$ satisfies (2.3.2) for some K and a, we say that a set is (K, a)-*homogeneous*, or simply *homogeneous*, and so the Assouad dimension of $\mathcal{A}$ is the infimum over all a such that $\mathcal{A}$ is (K, a)-homogeneous for some $K \geq 1$.

The idea of almost homogeneity is also of interest. A set $\mathcal{A}$ is said to be (α, β)-*almost* (K, a)*-homogeneous* or *almost homogeneous* if

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \leq K \left(\frac{r}{\rho}\right)^a \operatorname{slog}(r)^\alpha \operatorname{slog}(\rho)^\beta \tag{2.3.3}$$

for all $0 < \rho < r < \infty$, where slog is the *symmetric logarithm*, $\operatorname{slog}(x) = \log(x + x^{-1})$. Related to almost homogeneity is the (α, β)*-Assouad dimension of* $\mathcal{A}$, denote $\dim_A^{\alpha,\beta}$, and defined to be the infimum over all a such that $\mathcal{A}$ is (α, β)-almost (K, a)-homogeneous for some $K \geq 1$.

The concepts of homogeneity and almost-homogeneity play an important role in the embedding results of Section 2.4, which are a primary source of motivation for this thesis.

Note that there are examples of countable sets with Assouad dimension greater than zero. For instance, Mackay and Tyson [18, Example 1.4.15] give an example of a countable set with Assouad dimension 1, and Olson [22, Lemma 4.1] gives an example of a countable set with infinite Assouad dimension. In practical terms, this implies that the overall behavior of a system with respect to the Assouad dimension may be determined by a small part of that system.

While the Assouad dimension is difficult to compute, this sensitivity to the local complexity of a countable set makes it a useful tool in applications. Of particular interest for this thesis are embedding results that rely on the finiteness of $\dim_A(\mathcal{A}-\mathcal{A})$, as summarized in the following section.

2.4 Embedding Results

We begin this section with a definition.

Definition 2.4.1. Let $\mathcal{A} \subseteq \mathbb{R}^D$. The *set of differences* of $\mathcal{A}$, denoted $\mathcal{A} - \mathcal{A}$, is the set

$$\mathcal{A} - \mathcal{A} = \{x - y \mid x, y \in \mathcal{A}\}.$$

The set of differences is of interest, as it occurs naturally when embedding attractors into finite dimensional spaces. For example, an early result of Mañé [20] proves that if $\dim_H(\mathcal{A} - \mathcal{A}) < D$, then a residual set of projections continuously embed $\mathcal{A}$ into $\mathbb{R}^D$ (where $\dim_H$ denotes the Hausdorff dimension). Hunt and Kaloshin [11] show that if $\dim_f(\mathcal{A} - \mathcal{A}) < D$, then a prevalent set of projections continuously embed $\mathcal{A}$ into $\mathbb{R}^D$ with Hölder continuous inverse. Olson and Robinson [23] prove that if $\dim_A(\mathcal{A}-\mathcal{A}) < D$, then a prevalent set of projections continuously embed $\mathcal{A}$ into $\mathbb{R}^D$ with Lipschitz inverse, up to a logarithmic correction.

Here the terms residual and prevalent refer to notions of density. A residual set is dense in the sense that it is a set of Baire second category, while a prevalent set is dense in the sense of measure theory. Specifically, prevalence is a means of extending the usual notion of Lebesgue almost everywhere to infinite-dimensional spaces where translation invariant measures do not exist. Prevalence is defined by Hunt, Sauer, and Yorke [12] as follows.

Definition 2.4.2. Let V be a normed linear space. A Borel set $S \subseteq V$ is *shy* if there exists a compactly supported probability measure μ on V such that $\mu(S+v) = 0$ for every $v \in V$. More generally, a set is shy if it is contained in a shy Borel set. The complement of a shy set is said to be *prevalent*.

Another embedding result, due to Assouad [2], holds that the image of a homogenous set under a bi-Lipschitz map is homogenous, and under certain conditions, a homogenous subset of an infinite-dimensional Hilbert space may be embedded into a finite dimensional Euclidean space via a bi-Lipschitz map. One generalization of this result, due to Olson and Robinson [23], considers the class of almost bi-Lipschitz functions.

Definition 2.4.3. Let (X,d) and $(\tilde{X},\tilde{d})$ be metric spaces. A map $f:(X,d)\to(\tilde{X},\tilde{d})$ is said to be γ*-almost bi-Lipschitz*, or simply *almost bi-Lipschitz*, if there are $\gamma \geq 0$ and $L > 0$ such that

$$\frac{1}{L}\frac{d(x,y)}{\operatorname{slog}(d(x,y))^{\gamma}} \leq \tilde{d}(f(x),f(y)) \leq Ld(x,y)$$

for all $x,y \in X$ such that $x \neq y$. Recall that $\operatorname{slog}(x) = \log(x+x^{-1})$.

A function satisfying Definition 2.4.3 is bi-Lipschitz up to a logarithmic correction. The image of a homogenous set under an almost bi-Lipschitz map is not homogenous, but it is almost homogenous, as is the image of an almost homogenous map. This gives rise to the following generalization of Assouad's result:

Theorem 2.4.4 ([23, Theorem 5.6])**.** *Let $\mathcal{A}$ be a compact subset of a Hilbert space $\mathscr{H}$ such that $\mathcal{A}-\mathcal{A}$ is (α,β)-almost homogeneous with* $\dim_A^{\alpha,\beta}(\mathcal{A}-\mathcal{A}) < a < D$*. If*

$$\gamma > \frac{2 + D(3+\alpha+\beta) + 2(\alpha+\beta)}{2(D-a)}$$

then a prevalent set of linear maps $f\colon \mathscr{H} \to \mathbb{R}^D$ are injective on $\mathcal{A}$ and, in particular, γ-almost bi-Lipschitz.

Essentially, if $\mathcal{A}$ is the infinite dimensional attractor of a dynamical system, but $\dim_A(\mathcal{A}-\mathcal{A})$ is finite, then $\mathcal{A}$ may be embedded into a finite dimensional space while preserving the dynamics of the system. Similar results are given in [22, Theorem 5.2] and [24, Theorem 9.20].

It is then of interest whether or not knowledge of the dimension of $\mathcal{A}$ can give insight into the dimension of $\mathcal{A}-\mathcal{A}$. In particular, under what conditions does the finiteness of $\dim_A(\mathcal{A})$ ensure the finiteness of $\dim_A(\mathcal{A}-\mathcal{A})$?

For many notions of dimension, the finite dimensionality of $\mathcal{A}$ does not ensure that $\mathcal{A}-\mathcal{A}$ is of finite dimension. For example, there are sets with finite Hausdorff dimension which have sets of differences of infinite Hausdorff dimension. Fractal dimension is unusual among notions of dimension in that the inequality $\dim_f(\mathcal{A}-\mathcal{A}) \le 2\dim_f(\mathcal{A})$ always holds (see [22], [23], [24]). We might hope that a similar result holds for the Assouad dimension, but as noted in Chapter 5, there exist examples of sets $\mathcal{A}$ that satisfy no such bound.

Moreover, many of the results given in Chapter 5 relate to the construction of simple sets that occur as the attractors of iterated function systems of similarities which satisfy the Moran open set condition. These sets are highly structured, which suggests that the Assouad dimension of the set of the differences might satisfy some bound in terms of the Assouad dimension of the set itself. As we will show, even this structure is not sufficient to ensure that $\dim_A(\mathcal{A} - \mathcal{A}) \leq K \dim_A(\mathcal{A})$ for any constant K.

Chapter 3

Grid Like Systems

In this chapter, we address the question of whether the open set condition is sufficient to guarantee that $\dim_A(\mathcal{A}) = \dim_s(\mathcal{A})$ in the case where $\mathcal{A}$ is a self-similar fractal, originally posed in an email from Luukkainen in 2008 [15]. Affirmative answers were given in 2010 by Mackay [17] and in 2011 by Mackay and Tyson [18] using the theory of Ahlfors regular measures. We present an independent, elementary proof based on a concept which we have termed the grid like property, then extend our result to more general sets.

3.1 Similarities of Uniform Ratio

We begin with an result that avoids some technical details and gives a flavor for the general result. Consider the simplified setting of a self-similar fractal attractor of an iterated function system satisfying the open set condition in which each map has the same contraction ratio. We shall show in Proposition 3.1.2 that in such a setting, the similarity and Assouad dimensions coincide.

The strength of the open set condition is that it ensures that images of the Moran open set under distinct maps of such an iterated function system do not intersect, and, as the attractor is contained in the closure of the open set, images of the attractor do not overlap too much. The following lemma, which is special case of one of Hutchinson's results [13, 5.2.3(iii)], extends this disjointness property to arbitrary sequences of functions of fixed length.

Lemma 3.1.1. *Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system in $\mathbb{R}^D$ with Moran open set U. Suppose that the contraction ratio of f_i is $c \in (0,1)$ for all $i = 1, 2, \ldots, L$ and fix $n \in \mathbb{N}$. If $\alpha, \beta \in \mathcal{S}_L$, $\ell(\alpha) = \ell(\beta) = n$, and $\alpha \neq \beta$, then $f^{\alpha}(U) \cap f^{\beta}(U) = \emptyset$.*

Proof. Denote $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. Let k be the largest index such that $\alpha_i = \beta_i$ for all $i < k$ and note that $k \leq n$, as otherwise α and β would be equal.

By hypothesis,

$$f_{\alpha_{k+1}} \circ f_{\alpha_{k+2}} \circ \cdots \circ f_{\alpha_n}(U) \subseteq U \qquad \text{and} \qquad f_{\beta k+1} \circ f_{\beta k+2} \circ \cdots \circ f_{\beta_n}(U) \subseteq U.$$

This implies that

$$f_{\alpha_k} \circ (f_{\alpha_{k+1}} \circ f_{\alpha_{k+2}} \circ \cdots \circ f_{\alpha_m})(U) \cap f_{\beta_k} \circ (f_{\beta k+1} \circ f_{\beta k+2} \circ \cdots \circ f_{\beta_n})(U) = \emptyset.$$

As f_i is one-to-one for each $i = 1, 2, \ldots, L$, it follows that f^{γ} is one-to-one for any $\gamma \in \mathcal{S}_L$. As f^{γ} is one-to-one, if $V \cap W = \emptyset$, then $f^{\gamma}(V) \cap f^{\gamma}(W) = \emptyset$. In particular,

$$f^{\alpha} = (f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_{k-1}}) \circ f_{\alpha_k} \circ (f_{\alpha_{k+1}} \circ f_{\alpha_{k+2}} \circ \cdots \circ f_{\alpha_m})(U)$$

and

$$f^{\beta} = (f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_{k-1}}) \circ f_{\beta_k} \circ (f_{\beta k+1} \circ f_{\beta k+2} \circ \cdots \circ f_{\beta_n})(U)$$

are disjoint, which is the desired result. $\square$

With this result in hand, we can now prove the claimed equality.

Proposition 3.1.2. *Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system of similarities in $\mathbb{R}^D$ with Moran open set U. Let $\mathcal{A}$ be the invariant set of F, and suppose that the contraction ratio of f_i is $c \in (0,1)$ for each $i = 1, 2, \ldots, L$. Then $\dim_A(\mathcal{A}) = \dim_s(\mathcal{A})$.*

Proof. Choose $r, \rho \in \mathbb{R}$ such that $0 < \rho < r$ and let $p, q \in \mathcal{A}$ be arbitrary points. Let $\eta = \operatorname{diam}(\mathcal{A})$, and $\delta = \operatorname{diam}(U)$. Let $\nu = \lambda^D(U)$ be the D-dimensional volume of U, and $\Omega_D = \lambda^D(\mathrm{B}_1(0))$ be the D-dimensional volume of the unit ball.

If $r < \delta$, then there is some $m \in \mathbb{N}$ such that $c^m\delta < r \le c^{m-1}\delta$. Otherwise, take $m = 0$. In either case, let

$$A = \{\alpha \in \mathcal{S}_L \mid \ell(\alpha) = m, f^{\alpha}(\mathcal{A}) \cap \mathrm{B}_r(p) \neq \emptyset\}.$$

Let $N = (\Omega_D/\nu)(2\delta/c)^D$, a constant the does not depend on the choice of either r or p. Then $\operatorname{card}(A) \le N$. To see this, define

$$A^{\star} = \{\alpha \in \mathcal{S}_L \mid \ell(\alpha) = m, f^{\alpha}(\overline{U}) \cap \mathrm{B}_r(p) \neq \emptyset\}.$$

For any $\alpha \in \mathcal{S}_L$, the containment $f^{\alpha}(\mathcal{A}) \subseteq f^{\alpha}(\overline{U})$ holds, and so if $\alpha \in A$, then $\alpha \in A^{\star}$. Thus $A \subseteq A^{\star}$, which implies that $\operatorname{card}(A) \le \operatorname{card}(A^{\star})$.

To estimate $\operatorname{card}(A^{\star})$, note that if $\alpha, \beta \in A^{\star}$ are distinct, then by Lemma 3.1.1,

$f^\alpha(U) \cap f^\beta(U) = \emptyset$. Hence the volume of the union of images of U determined by $A^\star$ is given by

$$\lambda^D\left(\bigcup_{\alpha\in A^\star} f^\alpha(U)\right) = \sum_{\alpha\in A^\star} \lambda^D(f^\alpha(U)) = \sum_{\alpha\in A^\star} (c_\alpha)^D \nu = \sum_{\alpha\in A^\star} (c^m)^D \nu.$$

By definition of $A^\star$, $r/\delta \leq c_{\alpha'} = c^{m-1}$ for any $\alpha \in A^\star$. Thus

$$\sum_{\alpha\in A^\star} (c^m)^D \nu \geq \sum_{\alpha\in A^\star} \left(\frac{cr}{\delta}\right)^D \nu = N^\star \left(\frac{cr}{\delta}\right)^D \nu.$$

As $f^\alpha(\overline{U}) \cap \mathrm{B}_r(p) \neq \emptyset$ for each $\alpha \in A^\star$, and $\operatorname{diam}(f^\alpha(\overline{U})) = c^m\delta < r$, it holds that $f^\alpha(\overline{U}) \subseteq \mathrm{B}_{2r}(p)$. Thus

$$\begin{aligned} N^\star \left(\frac{cr}{\delta}\right)^D \nu &\leq \lambda^D\left(\bigcup_{\alpha\in A^\star} f^\alpha(U)\right) \leq \lambda^D\left(\bigcup_{\alpha\in A^\star} f^\alpha(\overline{U})\right) \\ &\leq \lambda^D(\mathrm{B}_{2r}(p)) = \Omega_D(2r)^D. \end{aligned}$$

Solving for $N^\star$, we obtain

$$N^\star \leq \frac{\Omega_D}{\nu}\left(\frac{2\delta}{c}\right)^D = N$$

Hence $\operatorname{card}(A) \leq \operatorname{card}(A^\star) = N^\star \leq N$. That is, the cardinality of A is bounded by N, a constant which does not depend on either r or p.

Now we append additional maps to each of the sequences in A in order to produce images of the attractor that are contained in a ball of radius ρ. As above, if $\rho < \delta$, then there is some $n \in \mathbb{N}$ such that $c^{m+n}\delta < \rho \leq c^{m+n-1}\delta$. Otherwise, take $n = 0$. In either case, take

$$B = \{\beta \in \mathcal{S}_L \mid \ell(\beta) = n\}.$$

Clearly, $\operatorname{card}(B) = L^n$. Moreover, note that

$$\bigcup_{\beta \in B} f^\beta(\mathcal{A}) = \mathcal{A},$$

and that $f^\alpha \circ f^\beta(\mathcal{A}) \subseteq \mathrm{B}_\rho(f^\alpha \circ f^\beta(q))$. Hence for each $\alpha \in A$, we have

$$f^\alpha(\mathcal{A}) = f^\alpha\left(\bigcup_{\beta \in B} f^\beta(\mathcal{A})\right) = \bigcup_{\beta \in B} f^\alpha \circ f^\beta(\mathcal{A}) \subseteq \bigcup_{\beta \in B} \mathrm{B}_\rho(f^\alpha \circ f^\beta(q)).$$

As $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A})$, we have

$$\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A}) \subseteq \bigcup_{\alpha \in A} \bigcup_{\beta \in B} \mathrm{B}_\rho(f^\alpha \circ f^\beta(q)).$$

It remains only to estimate the number of balls of radius ρ there are in the union:

$$\mathcal{N}_{\mathcal{A} \cap \mathrm{B}_r(p)}(\rho) = \sum_{\alpha \in A} \operatorname{card}(B) = \sum_{\alpha \in A} L^n \leq N L^n.$$

n was selected such that $c^{m+n}\delta < \rho$, hence $n < \log(\rho/c^m\delta)/\log(c)$; and m was selected such that $r \leq c^{m-1}\delta$, hence $1/c^m \leq c\delta/r$. Thus

$$\begin{aligned} NL^n < NL^{\log(\rho/c^m\delta)/\log(c)} &= N\left(\frac{\rho}{c^m\delta}\right)^{\log(L)/\log(c)} \leq N\left(\frac{\rho c\delta}{\delta r}\right)^{\log(L)/\log(c)} \\ &= Nc^{\log(L)/\log(c)}\left(\frac{\rho}{r}\right)^{\log(L)/\log(c)} = NL\left(\frac{r}{\rho}\right)^{-\log(L)/\log(c)}. \end{aligned}$$

But NL is constant with respect to r and ρ, hence it follows from (2.3.2) that $\dim_A(\mathcal{A}) \leq -\log(L)/\log(c) = \dim_s(\mathcal{A})$. Then by (2.3.1) and (2.2.3), we have that $\dim_s(\mathcal{A}) = \dim_f(\mathcal{A}) \leq \dim_A(\mathcal{A})$. Therefore $\dim_A(\mathcal{A}) = \dim_s(\mathcal{A})$. $\square$

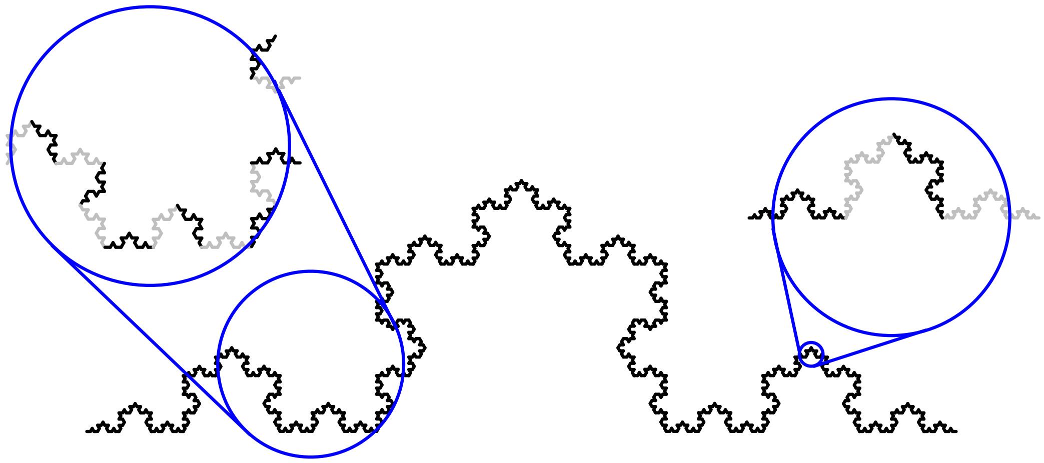

Figure 3.1: The Attractor of a Grid Like System

3.2 Grid Like Systems

Definition 3.2.1. Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system with attractor $\mathcal{A}$ in $\mathbb{R}^D$. F is said to be *grid like* if there exists $N \in \mathbb{N}$ such that for every $r > 0$ and any $p \in \mathbb{R}^D$, there is a set $A \subseteq \mathcal{S}_L$ such that

(1) $\operatorname{card} A \leq N$,

(2) $\operatorname{diam}(f^{\alpha}(\mathcal{A})) < r$ for each $\alpha \in A$, and

(3) $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^{\alpha}(\mathcal{A})$.

Intuitively, if an iterated function system is grid like, then there is some finite N such that any ball of radius r can be covered by a set of at most N images of the form $f^{\alpha}(\mathcal{A})$, where each such image is less than r in diameter. We illustrate this in Figure 3.2.1 with the von Koch curve (the von Koch curve is described in greater detail in, for example, [6, pages 18–20] or [8, page xviii]). The system whose attractor is the von Koch curve satisfies Moran's open set condition and, as we shall prove in Proposition 3.3.4, this implies that the system is grid like. For illustrative purposes, note that a ball of any radius r which intersects the von Koch curve intersects an unbounded number of images of the curve which are of diameter less than r. The intersection of the curve with the large ball has been covered by 14 images of the curve that have diameters smaller than the radius of the ball, while the smaller ball can be covered by four such images. It is possible to prove that any ball of radius r can be covered by $N = 16$ or fewer images of diameter r, and that this bound is sharp.

We now turn to a technical lemma which will shall use to prove Proposition 3.2.3.

Lemma 3.2.2. *For each $i \in \{1, 2, \ldots, L\}$, let $c_i \in \mathbb{R}$ be such that $0 < c_i < 1$ and let $s \in \mathbb{R}$ be such that $\sum_{i=1}^{L} (c_i)^s = 1$. Choose $0 < \sigma$ and let*

$$A_\sigma = \begin{cases} \{\alpha \in \mathcal{S}_L \mid c_\alpha < \sigma \leq c_{\alpha'}\} & \textit{if } \sigma \leq 1, \\ \{\phi\} & \textit{if } \sigma > 1. \end{cases}$$

Then $\sum_{\alpha \in A_\sigma} (c_\alpha)^s = 1$.

Recall that, as noted in Definition 2.2.1, ϕ is the empty sequence, $\ell(\phi) = 0$, f_ϕ is the identity function and $c_\phi = 1$.

Proof of Lemma 3.2.2. For any set of finite sequences A, let $M(A) = \max\{\ell(\alpha) \mid \alpha \in A\}$. Proof of the lemma is by induction on $M(A_\sigma)$. For the base case, suppose that $M(A_\sigma) = 0$. Then $A_\sigma = \{\phi\}$, from which it follows that

$$\sum_{\alpha \in A_\sigma} (c_\alpha)^s = (c_\phi)^s = 1^s = 1.$$

Hence the lemma holds when $M(A_\sigma) = 0$.

For induction, suppose that for some $n > 0$, if $M(A_\sigma) < n$, then the lemma holds.

Let A_{σ_0} be such that $M(A_{\sigma_0}) = n$, and for each $i \in \{1, 2, \ldots, L\}$, define

$$B_i = \{\beta \mid \alpha = (i, \beta) \text{ for some } \alpha \in A_{\sigma_0}\},$$

where, given a sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, the notation (i, γ) denotes the sequence $(i, \gamma_1, \gamma_2, \ldots, \gamma_m)$. Clearly, $M(B_i) = n - 1$. We claim that $B_i = A_{\sigma_0/c_i}$. To prove this claim, there are two cases to consider:

(1) Suppose that $\sigma_0/c_i \leq 1$. Then $\beta \in B_i$ if and only if $(i, \beta) \in A_{\sigma_0}$. By assumption, $\sigma_0 \leq c_i$, which implies that $\beta \neq \phi$, and so $c_{\beta'}$ is well-defined. By definition of A_{σ_0}, we have that $c_i c_\beta < \sigma_0 \leq c_i c_{\beta'}$, and so $c_\beta < \sigma_0/c_i \leq c_{\beta'}$. This inequality holds if and only if $\beta \in A_{\sigma_0/c_i}$. Thus $B_i = A_{\sigma_0/c_i}$.

(2) Suppose that $\sigma_0/c_i > 1$. $A_{\sigma_0/c_i} = \{\phi\}$ by definition. Let $\beta \in B_i$, and suppose for contradiction that $\beta \neq \phi$. Then $(i, \beta) \in A_{\sigma_0}$, which implies that $\sigma_0 < c_i c_{\beta'}$. But then $\sigma_0/c_i < c_{\beta'}$, and $c_{\beta'} \leq 1$. This contradicts the assumption that $\sigma_0/c_i > 1$, and so $\beta = \phi$. Thus $B_i = A_{\sigma_0/c_i}$.

In either case, we have that $B_i = A_{\sigma_0/c_i}$ as claimed. Thus B_i is a set of the form A_σ (where $\sigma = \sigma_0/c_i$) with $M(B_i) < n$. By the induction hypothesis, $\sum_{\beta \in B_i} (c_\beta)^s = 1$, from which it follows that

$$\sum_{\alpha \in A_\sigma} (c_\alpha)^s = \sum_{i=1}^{L} \left(c_i \sum_{\beta \in B_i} (c_\beta) \right)^s = \sum_{i=1}^{L} (c_i)^s = 1,$$

thus completing the proof. □

In the following proposition, note that s is defined so that if F is an iterated function system of similarities with attractor $\mathcal{A}$, then $\dim_s(\mathcal{A}) = s$. However, the proposition is stated more generally, as the result holds even when $\mathcal{A}$ is not self-similar and the similarity dimension is not defined.

Proposition 3.2.3. *Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system with contraction ratios c_i and the invariant set $\mathcal{A}$. Let $s \in \mathbb{R}$ be such that $\sum_{i=1}^{L} c_i^s = 1$. If F is grid like, then* $\dim_A(\mathcal{A}) \leq s$*.*

Proof. Choose $r, \rho \in \mathbb{R}$ such that $0 < \rho < r < 1$, and let $p, q \in \mathcal{A}$ be arbitrary points. Let $\eta = \operatorname{diam}(\mathcal{A})$, and $\eta_\alpha = \operatorname{diam}(f^\alpha(\mathcal{A}))$. Note that as the f_i are contractions with ratios $c_i < 1$, we have $\eta_\alpha \leq \eta c_\alpha$. As F is grid like, there is some $N \in \mathbb{R}$ and set of sequences A such that $\operatorname{card}(A) \leq N$, $\eta_\alpha < r$ for each $\alpha \in A$, and $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A})$. Take $K = N/c^s$, and note that K does not depend on either r or ρ.

For each $\alpha \in A$, let

$$B_\alpha = \{\beta \mid \eta_\alpha c_\beta < \rho \leq \eta_\alpha c_{\beta'}\}.$$

With $\sigma = \rho/\eta_\alpha$, it follows from Lemma 3.2.2 that $1 = \sum_{\beta \in B_\alpha} (c_\beta)^s$. As $\rho \leq c_\alpha c_{\beta'} \eta$, we have

$$1 = \sum_{\beta \in B_\alpha} (c_\beta)^s \geq \sum_{\beta \in B_\alpha} \left(\frac{\rho c}{\eta_\alpha}\right)^s.$$

Thus $\operatorname{card}(B_\alpha) \leq (\eta_\alpha/\rho c)^s$ for all $\alpha \in A$. For each $\beta \in B_\alpha$, it is clear that $f^\alpha \circ f^\beta(\mathcal{A}) \subseteq \mathrm{B}_\rho(f^\alpha \circ f^\beta(q))$. Moreover, $\bigcup_{\beta \in B_\alpha} f^\beta(\mathcal{A}) = \mathcal{A}$. Hence

$$f^\alpha(\mathcal{A}) = f^\alpha\left(\bigcup_{\beta \in B_\alpha} f^\beta(\mathcal{A})\right) = \bigcup_{\beta \in B_\alpha} f^\alpha \circ f^\beta(\mathcal{A}) \subseteq \bigcup_{\beta \in B_\alpha} \mathrm{B}_\rho(f^\alpha \circ f^\beta(q)).$$

From this, it follows that

$$\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} \bigcup_{\beta \in B_\alpha} \mathrm{B}_\rho(f^\alpha \circ f^\beta(q)).$$

The above is a union of ρ-balls which cover $\mathrm{B}_r(p)$. Thus the Assouad dimension of $\mathcal{A}$ is bounded by estimating the number of ρ-balls in the union. That is,

$$\begin{aligned}\mathcal{N}_{\mathcal{A}\cap \mathrm{B}_r(p)}(\rho) &= \sum_{\alpha\in A} \operatorname{card}(B_\alpha) \le \sum_{\alpha\in A} \left(\frac{\eta_\alpha}{\rho c}\right)^s \\ &\le \sum_{\alpha\in A} \left(\frac{r}{\rho c}\right)^s \le \frac{N}{c^s}\left(\frac{r}{\rho}\right)^s = K\left(\frac{r}{\rho}\right)^s\end{aligned}$$

Then by (2.3.2), $\dim_A(\mathcal{A}) \le s$. □

It is of note that this bound is identical in form to that given by Falconer [7, Theorem 8.8] for the Hausdorff dimension. In addition to providing an upper bound, the theorem states that for an iterated function system $F = \{f_i\}_{i=1}^L$ with constants d_i such that $d_i|x-y| \le |f_i(x) - f_i(y)|$ for all $x, y \in \mathbb{R}^D$, then the Hausdorff dimension of the attractor is bounded below by t such that $\sum_{i=1}^L (d_i)^t = 1$. It is possible to obtain a similar lower bound for the Assouad dimension of such an attractor, but the result is uninteresting, as the Hausdorff dimension provides a lower bound for the Assouad dimension (this follows from (2.3.1) and [8, Proposition 4.1]).

Moreover, as shall be demonstrated in Chapter 4, the upper bound given by Proposition 3.2.3 is not generally sharp. Before we examine such examples, we turn to the main results of this section, which give conditions sufficient to ensure that $\dim_f(\mathcal{A}) = \dim_A(\mathcal{A})$.

3.3 Assouad Dimension and Moran Open Sets

In Proposition 3.3.4, we will show that a system satisfying the open set condition is grid like. The structure of the proof is similar to that used to obtain a bound on $\operatorname{card}(A)$ in Proposition 3.1.2, though it is necessary to introduce some additional notation to handle differing contraction ratios, which is done in the following lemmas.

Definition 3.3.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m), \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{S}_L$. We say that α is a *prefix* of β if $m \le n$ and $\alpha_i = \beta_i$ for all $i = 1, 2, \ldots, m$.

Lemma 3.3.2. *For each $i = 1, 2, \ldots, L$, let $c_i \in (0,1)$. Choose $r < 1$, and let $A = \{\alpha \in \mathcal{S}_L \mid c_\alpha < r \le c_{\alpha'}\}$. If $\alpha, \beta \in A$ are distinct elements, then α is not a prefix of β and β is not a prefix of α.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be distinct elements of A. Without loss of generality, suppose that $m \le n$ (otherwise, relabel α and β), and let k be the largest index such that $\alpha_i = \beta_i$ for all $i \le k$. Clearly, if $m = n$, then $k < m$, as otherwise we would have $\alpha = \beta$. If $m < n$, assume for contradiction that $k = m$. As $\alpha, \beta \in A$, we have

$$r > c_\alpha = \prod_{i=1}^{k} c_{\alpha_i} = \prod_{i=1}^{k} c_{\beta_i} \ge \prod_{i=1}^{n-1} c_{\beta_i} = c_{\beta'} \ge r,$$

which is a contradiction. Hence $k < m$. Therefore α is not a prefix of β. By an identical argument, β is not a prefix of α. □

Suppose that U is a Moran open set for some iterated function system. Then under any two distinct maps in the system, the images of U are disjoint. Intuitively, we expect that images of U will also be disjoint under any two arbitrary sequences of maps, assuming that neither sequence is a prefix of the other. In other words, given two sequences of maps such that neither can be formed by appending terms to the other, the images of the attractor under each sequence of maps will be disjoint. While a similar result is proved by Hutchinson [13], we give an alternative proof for completeness.

Lemma 3.3.3. *Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system which satisfies the open set condition with a Moran open set U. Let $\alpha, \beta \in \mathcal{S}_L$ such that α is not a prefix of β and β is not a prefix of α. Then $f^{\alpha}(U) \cap f^{\beta}(U) = \emptyset$.*

Proof. As U is a Moran open set of F, the set U is nonempty, bounded, open, $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j$ and $f_i(U) \subseteq U$ for all $i = 1, 2, \ldots, L$. Denote $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, and let k be the largest index such that $\alpha_i = \beta_i$ for all $i < k$. By hypothesis,

$$f_{\alpha_{k+1}} \circ f_{\alpha_{k+2}} \circ \cdots \circ f_{\alpha_m}(U) \subseteq U \quad \text{and} \quad f_{\beta k+1} \circ f_{\beta k+2} \circ \cdots \circ f_{\beta_n}(U) \subseteq U.$$

This implies that

$$f_{\alpha_k} \circ (f_{\alpha_{k+1}} \circ f_{\alpha_{k+2}} \circ \cdots \circ f_{\alpha_m})(U) \cap f_{\beta_k} \circ (f_{\beta k+1} \circ f_{\beta k+2} \circ \cdots \circ f_{\beta_n})(U) = \emptyset.$$

As f_i is one-to-one for each $i = 1, 2, \ldots, L$, it follows that f^{γ} is one-to-one for any $\gamma \in \mathcal{S}_L$. As f^{γ} is one-to-one, if $V \cap W = \emptyset$, then $f^{\gamma}(V) \cap f^{\gamma}(W) = \emptyset$. In particular,

$$f^{\alpha} = (f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_{k-1}}) \circ f_{\alpha_k} \circ (f_{\alpha_{k+1}} \circ f_{\alpha_{k+2}} \circ \cdots \circ f_{\alpha_m})(U)$$

and

$$f^{\beta} = (f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_{k-1}}) \circ f_{\beta_k} \circ (f_{\beta k+1} \circ f_{\beta k+2} \circ \cdots \circ f_{\beta_n})(U)$$

are disjoint, which is the desired result. $\square$

We are now able to prove the main result of this section.

Proposition 3.3.4. *Let $F = \{f_i\}_{i=1}^{L}$ be a collection of contracting similarities which satisfies the open set condition. Then F is grid like.*

Proof. As noted above, every iterated function system determines a unique, nonempty, compact invariant set $\mathcal{A}$ satisfying (2.2.1). For each $i = 1, 2, \ldots, L$, let c_i be the contraction ratio of f_i, and take $c = \min\{c_1, c_2, \ldots, c_L\}$.

As F satisfies the open set condition, there is some nonempty, bounded, open set U such that $f_i(U) \cap f_j(U) = \emptyset$ if $i \neq j$, and $f_i(U) \subseteq U$ for all $i = 1, 2, \ldots, L$. Let $\nu = \lambda^D(U)$ denote the D-dimensional volume of U, let $\delta = \operatorname{diam}(\overline{U})$ where $\overline{U}$ is the closure of U, and let $\eta = \operatorname{diam}(\mathcal{A})$. Take

$$N = \frac{\Omega_D (2\delta)^D}{\nu c^D},$$

where $\Omega_D = \lambda^D(\mathrm{B}_1(0))$ is the D-dimensional volume of a unit ball. Choose $r > 0$ and $p \in \mathbb{R}^D$ arbitrarily. If $r > \eta$, then take $A = \{\phi\}$ and note that $\operatorname{diam}(f^\phi(\mathcal{A})) = \operatorname{diam}(\mathcal{A}) = \eta < r$, and $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \mathcal{A} = f^\phi(\mathcal{A})$. Then A is a set with the required properties, and $\operatorname{card}(A) = 1 < N$.

Otherwise, let

$$A = \{\alpha \in \mathcal{S}_L \mid c_\alpha \delta < r \leq c_{\alpha'} \delta, f^\alpha(\mathcal{A}) \cap \mathrm{B}_r(p) \neq \emptyset\}.$$

Note that $\operatorname{diam}(f^\alpha(\mathcal{A})) = c_\alpha \eta \leq c_\alpha \delta < r$ for all $\alpha \in A$ and $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A})$. Hence it remains only to estimate the cardinality of A.

To this end, let

$$A^\star = \{\alpha \in \mathcal{S}_L \mid c_\alpha \delta < r \leq c_{\alpha'} \delta, f^\alpha(\overline{U}) \cap \mathrm{B}_r(p) \neq \emptyset\},$$

and let $N^\star = \operatorname{card}(A^\star)$. Note that as $\mathcal{A} \subseteq \overline{U}$, it follows that $f^\alpha(\mathcal{A}) \subseteq f^\alpha(\overline{U})$ for each $\alpha \in A$, hence if $\alpha \in A$, then $\alpha \in A^\star$. Hence $A \subseteq A^\star$, and $\operatorname{card}(A) \leq \operatorname{card}(A^\star)$.

It follows from Lemmas 4.2.3 and 3.3.3 that $f^\alpha(U) \cap f^\beta(U) = \emptyset$ if α and β are distinct elements of $A^\star$. Hence the D-dimensional volume of $\bigcup_{\alpha\in A^\star} f^\alpha(U)$ is given by

$$\lambda^D\left(\bigcup_{\alpha\in A^\star} f^\alpha(U)\right) = \sum_{\alpha\in A^\star} \lambda^D(f^\alpha(U)) = \sum_{\alpha\in A^\star} (c_\alpha)^D \nu = \sum_{\alpha\in A^\star} (c_{\alpha'} c_{\alpha_m})^D \nu,$$

where α_m is the last term of α. From the definition of $A^\star$, it follows that $c_{\alpha'} \geq r/\delta$, and $c_{\alpha_m} \geq c$ for each $\alpha \in A^\star$. Thus

$$\sum_{\alpha\in A^\star} (c_{\alpha'} c_{\alpha_m})^D \nu > \sum_{\alpha\in A^\star} \left(\frac{rc}{\delta}\right)^D \nu = N^\star \left(\frac{rc}{\delta}\right)^D \nu.$$

For each $\alpha \in A^\star$, $f^\alpha(\overline{U}) \cap \mathrm{B}_r(p) \neq \emptyset$ and $\operatorname{diam}(f^\alpha(\overline{U})) = c_\alpha \delta < r$, thus $f^\alpha(\overline{U}) \subseteq \mathrm{B}_{2r}(p)$. Hence $\bigcup_{\alpha\in A^\star} f^\alpha(\overline{U}) \subseteq \mathrm{B}_{2r}(p)$. This implies that

$$\begin{aligned} N^\star \left(\frac{rc}{\delta}\right)^D \nu < \lambda^D\left(\bigcup_{\alpha\in A^\star} f^\alpha(U)\right) &\leq \lambda^D\left(\bigcup_{\alpha\in A^\star} f^\alpha(\overline{U})\right) \\ &\leq \lambda^D(\mathrm{B}_{2r}(p)) = \Omega_D(2r)^D. \end{aligned}$$

Solving for $N^\star$, we have

$$N^\star < \Omega_D(2r)^d \left(\frac{\delta^D}{(2rc)^D \nu}\right) = \frac{\Omega_D(2\delta)^D}{\nu c^D} = N.$$

Hence $\operatorname{card}(A) \leq \operatorname{card}(A^\star) = N^\star \leq N$.

Thus there is an N such that for every $r > 0$ and $p \in \mathbb{R}^D$, the cardinality of A is less than N. Therefore any iterated function system satisfying the open set condition with a self-similar invariant set is grid like. □

The following is an immediate consequence of this result.

Corollary 3.3.5. *Let* $F = \{f_i\}_{i=1}^{L}$ *be a system of contracting similarities which satisfies the open set condition, and let* $\mathcal{A}$ *be the attractor of* F. *Then* $\dim_A(\mathcal{A}) = \dim_f(\mathcal{A})$.

Proof. The system F satisfies the open set condition, thus by Proposition 3.3.4, F is grid like. It then follows from Proposition 3.2.3 that $\dim_A(\mathcal{A}) \leq \dim_s(\mathcal{A})$. By (2.2.3), we have $\dim_s(\mathcal{A}) = \dim_f(\mathcal{A})$. Finally, as $\mathcal{A}$ is compact, it follows from (2.3.1) that $\dim_f(\mathcal{A}) \leq \dim_A(\mathcal{A})$. Therefore $\dim_A(\mathcal{A}) = \dim_f(\mathcal{A})$. □

Chapter 4

Examples of Grid Like Systems

The previous chapter proves several results about grid like systems, but fails to address the utility of the definition. We noted that all systems which satisfy the Moran open set condition are grid like, which raises two questions. First, are there examples of grid like systems which do not satisfy the open set condition? Second, are there examples of systems which are not grid like?

In this chapter, we answer both questions in the affirmative by offering examples of grid like systems which do not satisfy the open set condition, as well as examples of systems that are not grid like. In doing so, we hope both to justify the definition, and to give some intuition as to the behavior of grid-like systems.

4.1 Simple Examples

Falconer [8, page 192] describes the open set condition as a condition which ensures that the components $f_i(\mathcal{A})$ of $\mathcal{A}$ do not overlap "too much." A system that is grid like satisfies a similar condition: a grid like system is one in which the images of $\mathcal{A}$

under the maps f_i either don't overlap, or if they do overlap, they do so in a "nice" way.

Example 4.1.1. The system $\{f_1, f_2\}$ with $f_1 = f_2$ does not satisfy the open set condition, as $f_1(U) = f_2(U)$ for any open set U (or, indeed, any set at all), and so the intersection cannot be empty. However, this system is grid like with $N = 1$. To see this, first note that the attractor is a single point q, the point fixed by either map. Then for any r and p, $\mathcal{A} \cap \mathrm{B}_r(p)$ is either empty, or a single point. In either case, let $A = \{(1)\}$, and note that $\operatorname{diam}(f^\alpha(\mathcal{A})) = \operatorname{diam}(f_1(\mathcal{A})) = \operatorname{diam}(q) = 0$ and $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A}) = f_1(\mathcal{A}) = \mathcal{A}$. Thus this system is grid like.

Example 4.1.2. A slightly more interesting example is the system $G = \{g_1, g_2, g_3\}$ in $\mathbb{R}$, where

$$g_1(x) = \frac{1}{3}x, \qquad g_2(x) = \frac{1}{3}x + \frac{2}{3}, \qquad \text{and} \qquad g_3(x) = g_1 \circ g_2(x) = \frac{1}{9}x + \frac{2}{9}.$$

Let $\mathcal{B}$ denote the attractor of G. Consider the subsystem $G^\star = \{g_1, g_2\}$. This subsystem, which is discussed in greater detail in Section 5.3, below, has the same attractor $\mathcal{B}$ as G. However, $G^\star$ satisfies the open set condition, and so by Proposition 3.3.4, it is grid like. Hence there is some $N \in \mathbb{N}$ such that for any $r > 0$ and $p \in \mathbb{R}^D$, there exists a set $A \subseteq \mathcal{S}_2$ with cardinality less than N such that $\operatorname{diam}(g_\alpha(\mathcal{B})) < r$ for each $\alpha \in A$ and $\mathcal{B} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} g_\alpha(\mathcal{B})$. But $\mathcal{S}_2 \subseteq \mathcal{S}_3$, from which it follows that G is also grid like.

At the heart of both of these examples is the idea that if two maps or compositions of maps are equal to each other, then one or the other is redundant, and may be removed from consideration. It is then sufficient to determine whether or not the system without the redundant maps is grid like.

4.2 A Grid Like System $\mathbb{R}^2$

Example 4.2.1. Consider the iterated function system $F = \{f_1, f_2, f_3, f_4\}$ where the maps $f_1, f_2, f_3, f_4 \colon \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x} - \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad f_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$f_3(\mathbf{x}) = \frac{1}{2}R_{\pi/2}\mathbf{x}, \qquad f_4(\mathbf{x}) = \frac{1}{4}\mathbf{x},$$

where $R_{\pi/2}$ is the matrix given by

$$R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $\mathcal{A}$ denote the attractor of the system F. An approximation of $\mathcal{A}$ is shown in Figure 4.1. That the set shown in the figure approximates $\mathcal{A}$ is a consequence of the following alternative characterization of the attractor, given by Falconer.

Theorem 4.2.2. *[8, Theorem 9.1] Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system with attractor $\mathcal{A}$. Let $\mathcal{K}$ denote the class of non-empty compact subsets of $\mathbb{R}^D$. For each set $K \in \mathcal{K}$, define the map*

$$F(K) = \bigcup_{i=1}^{L} f_i(K),$$

and let F^p denote the p-fold composition of F with itself (as in Section 2.2). Then

$$\mathcal{A} = \bigcap_{p=0}^{\infty} F^p(K)$$

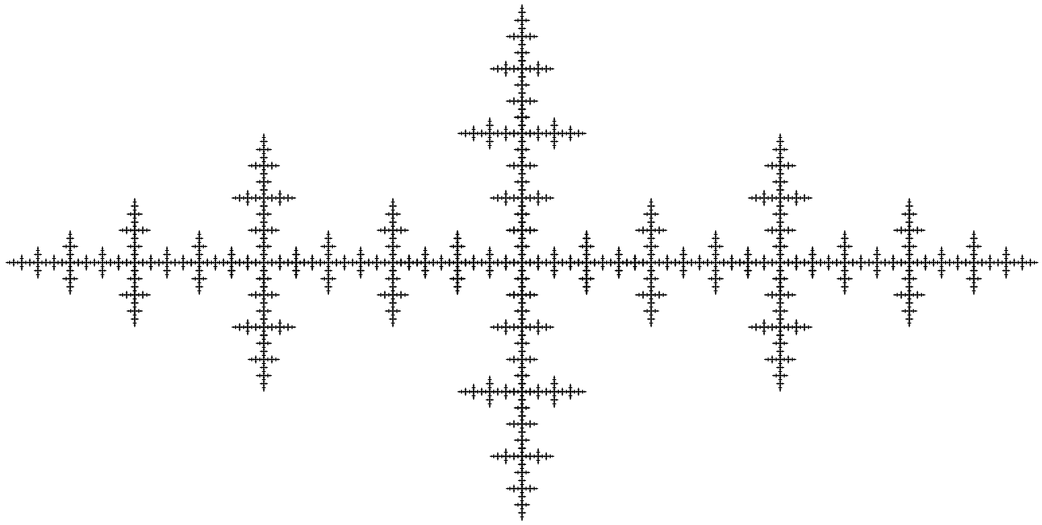

Figure 4.1: An Approximation of the Attractor $\mathcal{A}$

for every set $K \in \mathcal{K}$ such that $f_i(K) \subseteq K$ for all i.

Thus if K is such a set, the attractor $\mathcal{A}$ is contained in $F^p(K)$ for any $p \in \mathbb{N}$. Hence $\mathcal{A}$ can be approximated by finding a compact set K such that $F(K) \subseteq K$, and iterating the system forward on that set p times. Figure 4.1 is $F^{10}(\overline{\mathrm{B}_1(\mathbf{0})})$, where we claim that $F(\overline{\mathrm{B}_1(\mathbf{0})}) \subseteq \overline{\mathrm{B}_1(\mathbf{0})}$.

Lemma 4.2.3. *Let F and $\mathcal{A}$ be as above. Then $\mathcal{A} \subseteq \overline{\mathrm{B}_1(\mathbf{0})}$.*

Proof. Note that $\overline{\mathrm{B}_1(\mathbf{0})}$ is closed and bounded, therefore compact. Moreover,

$$f_1(\overline{\mathrm{B}_1(\mathbf{0})}) = \overline{\mathrm{B}_{1/2}(-\tfrac{1}{2}, 0)}, \quad f_2(\overline{\mathrm{B}_1(\mathbf{0})}) = \overline{\mathrm{B}_{1/2}(\tfrac{1}{2}, 0)}, \quad \text{and} \quad f_3(\overline{\mathrm{B}_1(\mathbf{0})}) = \overline{\mathrm{B}_{1/2}(\mathbf{0})},$$

where all three of the image balls are contained in $\overline{\mathrm{B}_1(\mathbf{0})}$. It then follows from Theorem 4.2.2 that

$$\mathcal{A} = \bigcap_{p=0}^{\infty} F^p(\overline{\mathrm{B}_1(\mathbf{0})}) \subseteq \overline{\mathrm{B}_1(\mathbf{0})},$$

which is the desired result. $\square$

In addition to providing a useful tool for generating images that approximate the attractor, this containment makes it possible to bound the diameter of the attractor. This bound then allows us to prove that F is grid like, which is the content of the following proposition.

Proposition 4.2.4. *F, as given above, is grid like.*

Proof. We claim that $N = 128$ satisfies Definition 3.2.1. By Lemma 4.2.3, $\mathcal{A} \subseteq \mathrm{B}_1(\mathbf{0})$, and so $\mathrm{diam}(\mathcal{A}) \leq 2$. Note that the points $\mathbf{p}_1 = (-1, 0)$ and $\mathbf{p}_2 = (1, 0)$ are fixed by the maps f_1 and f_2, respectively. That is $\mathbf{p}_1 = f_1(\mathbf{p}_1)$ and $\mathbf{p}_2 = f_2(\mathbf{p}_2)$. Thus $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{A}$, and so $\mathrm{diam}(\mathcal{A}) \geq \mathrm{dist}(\mathbf{p}_1, \mathbf{p}_2) = 2$. Hence $\mathrm{diam}(\mathcal{A}) = 2$.

Let $r \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^2$ be arbitrary. If $r \geq 1$, take A to be the set of all two term sequences in $\mathcal{S}_4$. Then $\mathrm{card}(A) = 16 < 128 = N$, $\mathrm{diam}(f^\alpha(\mathcal{A})) = 1/2 < r$ for all $\alpha \in A$, and $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \mathcal{A} = \bigcup_{\alpha \in A} f^\alpha(\mathcal{A})$.

Now suppose that $r < 1$. First, note that

$$f_4(\mathcal{A}) \subseteq f_1 \circ f_2(\mathcal{A}) \cup f_2 \circ f_1(\mathcal{A}).$$

Then for any $\alpha \in \mathcal{S}_4$, it is possible to construct a set of sequences $A \subseteq \mathcal{S}_3$ such that if $\beta \in A$ then $\mathrm{diam}(f^\alpha(\mathcal{A})) = \mathrm{diam}(f^\beta(\mathcal{A})$, and $f^\alpha(\mathcal{A}) \subseteq \bigcup_{\beta \in A} f^\beta(\mathcal{A})$. As any mapping f_4 can be reduced to combinations of mappings by f_1 and f_2, we need only consider sequences f^α where $\alpha \in \mathcal{S}_3$.

Let $Q_j \subseteq [-1, 1]^2$ denote the square regions in Figure 4.2, and take $m \in \mathbb{N}$ to be the unique number such that $(1/2)^m < r \leq (1/2)^{m-1}$. Define

$$A^\star = \bigcup_{j=1}^{4} \{\alpha \in \mathcal{S}_3 \mid \ell(\alpha) = m + 1, f^\alpha(Q_j) \cap \mathrm{B}_r(\mathbf{p}) \neq \emptyset\}.$$

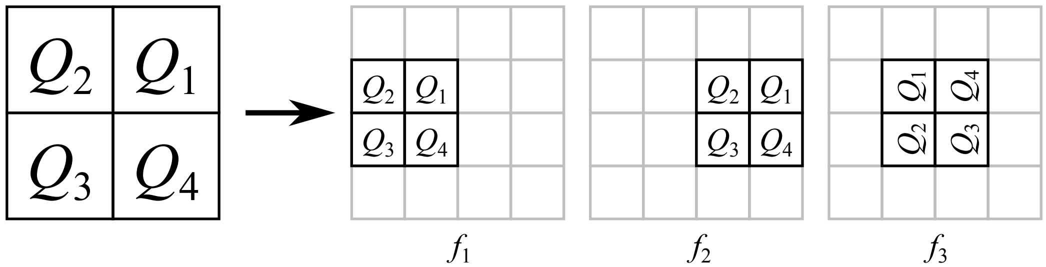

Figure 4.2: Images of $[-1,1]^2$ Under f_1, f_2, and f_3

For any $\alpha \in A^\star$ and any $j = 1, 2, 3, 4$, the set $f^\alpha(Q_j)$, as a set, is a square of the form

$$S^m_{xy} = \left[x\left(\frac{1}{2}\right)^{m+1}, (x+1)\left(\frac{1}{2}\right)^{m+1}\right] \times \left[y\left(\frac{1}{2}\right)^{m+1}, (y+1)\left(\frac{1}{2}\right)^{m+1}\right]$$

for some $x_\beta, y_\beta \in \mathbb{Z}$. Additionally, $f^\alpha(Q_j)$ is geometrically similar to Q_j, differing only in scale and by a possible rotation of 0, $\pi/2$, π, or $3\pi/4$ radians (refer to Figure 4.2). But then $\mathcal{A} \cap S^m_{xy}$ can be covered by 16 or fewer sets of the form $f^\alpha(Q_j \cap \mathcal{A})$, where $\alpha \in A^\star$. Moreover, note that as $r \leq (1/2)^{m-1}$, a ball of radius r intersects at most 16 squares of the form S^m_{xy}. Hence $\mathcal{A} \cap \mathrm{B}_r(p)$ can be covered by 128 or fewer images of the form $f^\alpha(\mathcal{A})$ where $\alpha \in A^\star$. Let $A \subseteq A^*$ denote this set of sequences of maps, and note that $\mathrm{card}(A) \leq 128$.

Finally, for any $\alpha \in A$,

$$\mathrm{diam}(f^\alpha(\mathcal{A})) = \mathrm{diam}(\mathcal{A})\left(\frac{1}{2}\right)^{m+1} = \left(\frac{1}{2}\right)^m < r.$$

Hence for any $r > 0$, there is a set $A \subseteq \mathcal{S}_3 \subseteq \mathcal{S}_4$ such that $\mathrm{card}(A) \leq N$, $\mathrm{diam}(f^\alpha(\mathcal{A})) < r$ for each $\alpha \in A$, and $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A})$. Therefore F is grid like. $\square$

It follows from Proposition 3.2.3 that $\dim_A(\mathcal{A}) \leq -\log(\frac{1}{2}(\sqrt{13}-3))/\log(2) \approx$

1.724. It should be noted that while F itself does not satisfy the open set condition, the system obtained by eliminating f_4 does satisfy the open set condition, and has the same attractor $\mathcal{A}$. An exact value for the Assouad dimension of $\mathcal{A}$ can then be computed by application of Corollary 3.3.5, which renders $\dim_A(\mathcal{A}) = \log(1-\sqrt{3})/\log(2) \approx 1.585$.

While the preceding paragraph demonstrates that it is possible to compute the Assouad dimension of $\mathcal{A}$ directly using the fact that a self-similar attractor of a system satisfying the open set condition is grid like, the primary motivation for the example was to show a direct proof that a system is grid like. Moreover, the precise relation between grid like systems and systems satisfying the open set condition is currently unresolved. For details, refer to Question 6.2.2.

4.3 Systems in $\mathbb{R}^2$ Which are Not Grid Like

In this section, we present a family of systems in $\mathbb{R}^2$ which are not grid like.

Example 4.3.1. Let $F = \{f_1, f_2, f_3\}$ be the iterated function system in $\mathbb{R}^2$ with maps

$$f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x} - \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad f_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad \text{and} \qquad f_3(\mathbf{x}) = \frac{1}{2}R_\theta\mathbf{x}, \tag{4.3.1}$$

where R_θ is the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

An approximation of the attractor $\mathcal{A}$ of this system is shown in Figure 4.3 for $\theta = 2\pi/(1+\sqrt{5})$. This image is $F^{10}(\overline{\mathrm{B}_1(\mathbf{0})})$, where an argument identical to that

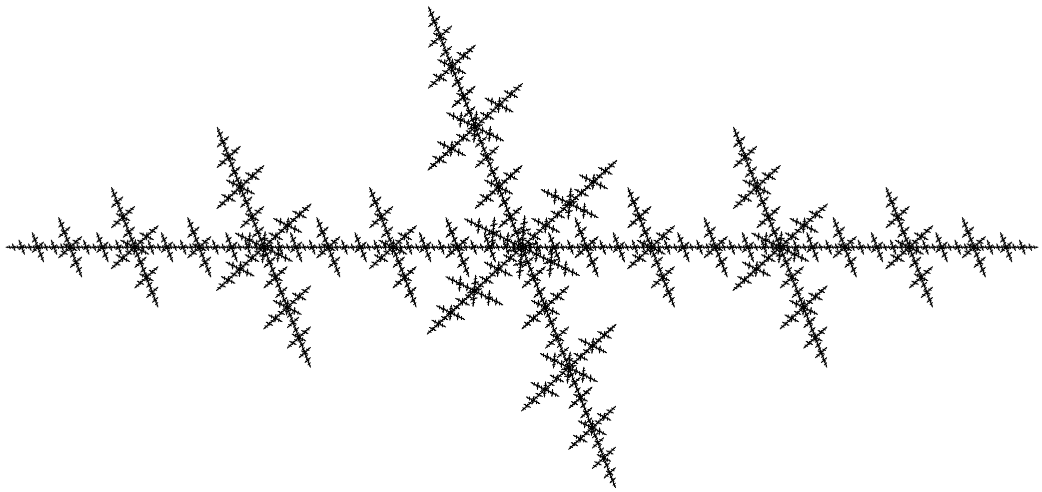

Figure 4.3: An Approximation of the Attractor of F for $\theta = \frac{2\pi}{1+\sqrt{5}}$

given in Lemma 4.2.3 shows that $\overline{\mathrm{B}_1(\mathbf{0})}$ is a compact set of the kind described in Theorem 4.2.2.

We claim that $\mathcal{A}$ is not grid like whenever θ is incommensurable with π. It is possible to show this directly, however we may use the results of Chapter 3 to prove the result in a somewhat simpler manner. In particular, it can be shown that $\dim_A(\mathcal{A}) = 2$ whenever θ is incommensurable with π, which is a contradiction to Proposition 3.2.3.

We first introduce the following notation:

Definition 4.3.2. Given a set $E \subseteq [0, 2\pi]$, let $\delta E = \sup\{d(x, E) \mid x \in [0, 2\pi]\}$, where $d(x, E)$ is the distance from a point x to the set E.

Of interest in the current context is the following elementary result, presented here for completeness.

Lemma 4.3.3. *Suppose that θ is incommensurable with π, that is, $\theta = \eta\pi$ where $\eta \in \mathbb{R} \setminus \mathbb{Q}$. For each $n \in \mathbb{N}$, define $E_n = \{i\theta (\mathrm{mod} 2\pi) \mid i = 0, 1, \ldots, n\}$. Then $\lim_{n\to\infty} \delta E_n = 0$.*

Proof. Let $\varepsilon > 0$ and choose $m > \frac{\pi}{4\varepsilon}$. Consider the collection of intervals of the form

$$\left[\frac{2k\pi}{m}, \frac{2(k+1)\pi}{m}\right],$$

with $k = 0, 1, \dots, m-1$. There are m such intervals, and as θ is incommensurable with π, the set $\{i\theta(\text{mod}2\pi) \mid i = 0, 1, \dots, m\} \subseteq [0, 2\pi]$ contains $m+1$ points. Thus by the Pigeonhole Principle there are distinct $i, j \in \{0, 1, \dots, m\}$ such that

$$i\theta(\text{mod}2\pi), j\theta(\text{mod}2\pi) \in \left[\frac{2k\pi}{m}, \frac{2(k+1)\pi}{m}\right]$$

for some $k \in \{0, 1, \dots, m-1\}$.

Note that $|i-j|\theta(\text{mod}2\pi) \in [0, \frac{2\pi}{m}]$, which implies that

$$|i-j|\theta(\text{mod}2\pi) \leq \frac{2\pi}{m} < 2\varepsilon.$$

By the Archimedean Principle, there is an L large enough that each $x \in [0, 2\pi]$ is contained in an interval of the form $[\ell|i-j|\theta(\text{mod}2\pi), (\ell+1)|i-j|\theta(\text{mod}2\pi)]$ for some $\ell \in \{0, 1, \dots, L-1\}$. Thus for each $x \in [0, 2\pi]$, there is an $\ell < L$ such that $|x - \ell|i-j|\theta(\text{mod}2\pi)| < \varepsilon$. Hence

$$\delta\{\ell|i-j|\theta(\text{mod}2\pi) \mid \ell = 0, 1, \dots, L-1\} \leq \varepsilon.$$

Let $N = L|i-j| \in \mathbb{N}$, and note that for all $n \geq N$, we have that

$$\{\ell|i-j|\theta(\text{mod}2\pi) \mid \ell = 0, 1, \dots, L-1\} \subseteq E_n.$$

It then follows that for all $n \geq N$, we have $\delta E_n \leq \varepsilon$, which completes the proof. $\square$

Lemma 4.3.4. *Let F and $\mathcal{A}$ be as above. The interval $I = [-1, 1] \times \{0\}$ is contained in $\mathcal{A}$.*

Proof. Suppose that K is nonempty a compact set such that $I \subseteq K$ and $F(K) \subseteq K$. Note that $f_1(I) \cup f_2(I) = I$, which implies that $I \subseteq F(I)$. As $I \subseteq K$, we have $F^n(I) \subseteq F^n(K)$ for all n, which implies that $I \subseteq F^n(K)$ for all n. Hence by Theorem 4.2.2, $I \subseteq \bigcap_{p=0}^{\infty} F^p(K) = \mathcal{A}$. Therefore $I \subseteq \mathcal{A}$, as claimed. □

As $I \subseteq \mathcal{A}$, so too are all of its images under sequences of maps from F. In particular, $f_3^n(I) \subseteq \mathcal{A}$ for any $n \in \mathbb{N}_0$. This collection of images has Assouad dimension 2, as is shown in the following lemma.

Lemma 4.3.5. *Let F, $\mathcal{A}$, and I be as in Lemma 4.3.4. Define*

$$\mathcal{I} = \bigcup_{k=1}^{\infty} f_3^k(I).$$

Then $\dim_A(\mathcal{I}) = 2$.

Proof. Suppose for contradiction that there is $\delta > 0$ such that $\dim_A(\mathcal{I}) = 2 - \delta$. Fix $\varepsilon > 0$. With E_n as in Lemma 4.3.3, there is a k large enough that $\delta E_k < \varepsilon$. Set $r = \frac{1}{2^k}$ and $\rho = \frac{\varepsilon}{2^k}$.

Choose a collection of points $\{\mathbf{p}_i\}_{i=1}^m \subseteq \mathbb{R}^2$ such that

$$\mathrm{B}_r(\mathbf{0}) \cap \mathcal{I} \subseteq \bigcup_{i=1}^{m} \mathrm{B}_\rho(\mathbf{p}_i).$$

By the choice of k, we have

$$\mathrm{B}_r(\mathbf{0}) \subseteq \bigcup_{i=1}^{m} \mathrm{B}_{2\rho}(\mathbf{p}_i).$$

To see this, note that $\mathcal{I}$ is made up of copies of $[0,1]\times\{0\}$, rotated by integer multiples of θ and scaled by factors of 2^{-n}. Intersecting these rotated intervals with $\mathrm{B}_r(\mathbf{0})$, we obtain $k+1$ intervals that coincide with diameters of the ball. By the choice of k, the maximum angle containing no such interval is at most ε. As such, each point in the ball is a distance of less than $\frac{\varepsilon}{2^k} = \rho$ from $\mathcal{I}$. Thus by doubling the radius of each ρ ball, it can be seen that $\mathrm{B}_r(\mathbf{0})$ can be covered by balls of radius 2ρ centered at the points $\{\mathbf{p}_i\}_{i=1}^m$.

Using volume estimates, we compute a lower bound for m, the number of ρ-balls required to cover an r-ball in $\mathcal{A}$, as follows:

$$\frac{\pi}{2^{2k}} = \pi r^2 = \lambda(\mathrm{B}_r(\mathbf{0}) \leq \lambda\left(\bigcup_{i=1}^{m} \mathrm{B}_{2\rho}(\mathbf{p}_i)\right) = m(\pi(2\rho)^2) = \frac{4m\pi\varepsilon}{2^k},$$

which, when solved for m, renders $m \geq \frac{1}{4\varepsilon}$. Hence

$$\mathcal{N}_{\mathcal{I}}\left(\frac{1}{2^k}, \frac{\varepsilon}{2^k}\right) \geq \frac{1}{4\varepsilon}.$$

By the characterization of Assouad dimension given in (2.3.2), there is some K such that for any choice of ε,

$$\mathcal{N}_{\mathcal{I}}\left(\frac{1}{2^k}, \frac{\varepsilon}{2^k}\right) \leq K\left(\frac{1}{\varepsilon}\right)^{2-\delta}.$$

Thus for any $\varepsilon > 0$, we have

$$K\left(\frac{1}{\varepsilon}\right)^{2-\delta} \geq \frac{1}{4\varepsilon}.$$

This implies that there is a constant $K^\star = (4K)^{1/\delta}$ such that for any $\varepsilon > 0$

$$\frac{1}{\varepsilon} \leq K^\star.$$

This is a contradiction, as $1/\varepsilon$ increases without bound as ε tends to zero. Thus no such $K^\star$ exists, hence the assumption $\dim_A(\mathcal{I}) < 2$ is false. Therefore $\dim_A(\mathcal{I}) \geq 2$.

Then, as $\mathcal{I} \subseteq \mathcal{A} \subseteq \mathbb{R}^2$, we have

$$2 \leq \dim_A(\mathcal{I}) \leq \dim_A(\mathcal{A}) \leq \dim_A(\mathbb{R}^2) = 2.$$

Therefore $\dim_A(\mathcal{A}) = 2$. □

Proposition 4.3.6. *Let F and $\mathcal{A}$ be as above. Then $\mathcal{A}$ is not grid like.*

Proof. Suppose for contradiction that F is grid like. It then follows from Proposition 3.2.3 that

$$\dim_A(\mathcal{A}) \leq \dim_s(\mathcal{A}) = \frac{\log(3)}{\log(2)}.$$

Thus $\dim_A(\mathcal{A}) < 2$.

By Lemma 4.3.5, $\mathcal{A}$ contains a set of Assouad dimension 2, and so $\dim_A(\mathcal{A}) \geq 2$. On the other hand, $\mathcal{A} \subseteq \mathbb{R}^2$, and the Assouad dimension of $\mathbb{R}^2$ is 2. Thus $\dim_A(\mathcal{A}) \leq 2$. Hence $\dim_A(\mathcal{A}) = 2$, which is a contradiction. Therefore F is not grid like. □

Chapter 5

Sets of Differences

When $\mathcal{A}$ is the attractor of an iterated function system, the Assouad dimension of $\mathcal{A} - \mathcal{A}$ is a question of interest. In particular, given $\mathcal{A}$, can the Assouad dimension of $\mathcal{A} - \mathcal{A}$ be bounded in any way as a function of the Assouad dimension of $\mathcal{A}$? In general, the answer is no.

For instance, Olson and Robinson [23, Proposition 8.3] demonstrate that if $\mathcal{A}$ is any connected subset of a Hilbert space $\mathscr{H}$ containing more than one point, then there exists a C^∞ bi-Lipschitz map $\phi : \mathscr{H} \to \mathscr{H}$ such that

$$\dim_A^{\alpha,\beta}(\phi(\mathcal{A}) - \phi(\mathcal{A})) = +\infty$$

for every $\alpha, \beta \geq 0$, where $\dim_A^{\alpha,\beta}$ is the (α, β)-Assouad dimension as given in Section 2.3. Moreover, ϕ may be chosen such that $\mathrm{dist}_{\mathscr{H}}(\phi(\mathcal{A}), \mathcal{A})$ is arbitrarily small.

Of particular interest is the case that $\mathcal{A}$ is the connected attractor of a dynamical system. This result shows that if $\mathcal{A}$ contains more than one point, it is always possible to find an arbitrarily small perturbation $\mathcal{A}^\varepsilon$ of $\mathcal{A}$ with the same dimension and dynamics of $\mathcal{A}$, but which ensures that the Assouad dimension of $\mathcal{A}^\varepsilon - \mathcal{A}^\varepsilon$ is infinite.

This result is abstract and gives little insight into the exact nature of such a perturbation. More recently, Eden, Kalaranov, and Zelik [5] have provided concrete results that are related to the current work. In their paper, they construct dynamical systems with compact attractors that cannot be embedded into any finite-dimensional log-Lipschitz manifold. It is likely that their techniques could be combined with the methods used in this section to construct concrete sets $\mathcal{A} \subseteq \mathscr{H}$ with $\dim_A(\mathcal{A})$ arbitrarily small and $\dim_A(\mathcal{A} - \mathcal{A}) = +\infty$. See Question 6.2.6 for more details.

In both of the above cited works, results are obtained by exploiting irregularities in abstract spaces and artificially constructed sets. *A priori*, it might be expected that a bound of the form $\dim_A(\mathcal{A} - \mathcal{A}) \leq K \dim_A(\mathcal{A})$ might be obtainable in the case that $\mathcal{A}$ carries more structure.

The self-similar attractors of iterated function systems which satisfy the open set condition are the most structured and regular sets that can be constructed, in the following sense: if $\mathcal{A}$ is such a set, then

$$\dim_L(\mathcal{A}) = \dim(\mathcal{A}) = \dim_H(\mathcal{A}) = \underline{\dim}_B = \dim_f = \dim_A(\mathcal{A}) = \dim_s(\mathcal{A}), \qquad (5.0.1)$$

where $\dim_L$ is the lower dimension (introduced by Larman [14] and studied by Fraser [9] as a dual to the Assouad dimension), dim is the topological dimension, $\dim_H$ is the Hausdorff dimension, $\underline{\dim}_B$ and $\dim_f$ are the lower-box counting and fractal (upper-box counting) dimensions (see [6], [7], [8], Definition 2.2.2), and $\dim_A$ and $\dim_s(\mathcal{A})$ are the Assouad and similarity dimensions as defined in Chapter 2. In general, the similarity dimension is not defined, and at least one of the remaining equalities in (5.0.1) is an inequality, with the smaller terms to the left (in fact, Luukkainen [16] defines a fractal as a set for which at least one of the above is an inequality). Thus the equality in (5.0.1) represents an special kind of structure.

This highly regular structure makes self-similar sets tractable. Examples of self-similar sets can easily be constructed, and their analysis is generally straighforward and simple. Moreover, self-similar sets occur quite naturally. Mandelbrot [19] asserts that many natural phenomena, such as the branching of trees and river networks, the distribution of islands in archipelagos, and the shape of coastlines, have self-similar structures corresponding to self-similar fractal sets.

Thus self-similar sets are of practical interest as they occur naturally, they are analytically tractable, and they contain sufficient structure that we might expect them to obey a bound of the kind described above. As we demonstrate in this chapter, even if $\mathcal{A}$ is the highly regular and structured self-similar attractor of an iterated function system satifying the open set condition, it is possible for $\dim_A(\mathcal{A})$ to be arbitrarily small, and $\dim_A(\mathcal{A} - \mathcal{A})$ to be maximal.

5.1 A System in $\mathbb{R}^2$

Consider the iterated function system $F = \{f_1, f_2\}$ in $\mathbb{R}^2$ with maps given by

$$f_1(\mathbf{x}) = c\mathbf{x} + \mathbf{b}_1 \qquad \text{and} \qquad f_2(\mathbf{x}) = cR_\theta\mathbf{x} + \mathbf{b}_2, \tag{5.1.1}$$

where $c \in (0, 1)$, R_θ is the matrix which sends points through a rotation of θ radians, and $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$ are distinct translations, i.e. $\mathbf{b}_1 \neq \mathbf{b}_2$. Denote the attractor of F by $\mathcal{A}$, and assume that F satisfies the Moran open set condition, so that $\dim_A(\mathcal{A}) = \log(2)/\log(\frac{1}{c})$. Note that the dimension of $\mathcal{A}$ can be made arbitrarily small by choosing sufficiently small c. However, if θ is incommensurable with π, that is, if $\theta = \eta\pi$ for some irrational number η, then $\dim_A(\mathcal{A} - \mathcal{A}) \geq 1$. This result is proved in Proposition 5.1.2. Before that proposition can be proved, we require a couple of technical results.

The Assouad dimension behaves nicely with respect to subsets, thus when computing lower bounds for the Assouad dimension of a set, it is often useful to consider the Assouad dimension of some distinguished, generally countable, subset. In the following lemma, we isolate a subset of $\mathcal{A} - \mathcal{A}$ for which the Assouad dimension can be computed exactly.

Lemma 5.1.1. *Let F and $\mathcal{A}$ be as above. Then for any $\mathbf{z} \in \mathcal{A} - \mathcal{A}$, the following containment holds:*

$$\bigcup_{k\in\mathbb{N}_0} \bigcup_{j=0}^{k} \{c^k R_{j\theta}\mathbf{z}\} \subseteq \mathcal{A} - \mathcal{A}$$

Proof. Let $\mathbf{z} \in \mathcal{A} - \mathcal{A}$. Then there exist $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{A} - \mathcal{A}$ such that $\mathbf{z} = \mathbf{x}_0 - \mathbf{x}_1$. As $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{A}$, it follows that $f^\alpha(\mathbf{x}_i) \in \mathcal{A}$ for any $\alpha \in \mathcal{S}_2$ and so $f^\alpha(\mathbf{x}_1) - f^\beta(\mathbf{x}_2) \in \mathcal{A} - \mathcal{A}$ for all $\alpha, \beta \in \mathcal{S}_2$. In particular, note that for any $k \in \mathbb{N}$ and any $j = 0, 1, \ldots, k$,

$$f_2^j \circ f_1^{k-j}(\mathbf{x}_i) = f_2^j\left(c^{k-j}\mathbf{x}_i + \sum_{l=0}^{k-j} c^l \mathbf{b}_1\right) = c^k R_{j\theta}\mathbf{x}_i + c^j \sum_{l=0}^{k-j} c^l \mathbf{b}_1 + \sum_{l=0}^{j} c^l \mathbf{b}_2.$$

Therefore

$$\mathcal{A} - \mathcal{A} \ni f_2^j \circ f_1^{k-j}(\mathbf{x}_0) - f_2^j \circ f_1^{k-j}(\mathbf{x}_1) = c^k R_{j\theta}(\mathbf{x}_0 - \mathbf{x}_1) = c^k R_{j\theta}(\mathbf{z}),$$

which completes the proof of the lemma. $\square$

We claim that this subset of $\mathcal{A} - \mathcal{A}$ has Assouad dimension 1. The proof is similar to that presented in Lemma 4.3.5, except that here we are concerned only with a set that is comparable to a circle, rather than a ball.

Proposition 5.1.2. *Let $\mathcal{A}$ be the attractor of $F = \{f_1, f_2\}$, where f_1 and f_2 are as given in (5.1.1). Suppose that θ is incommensurable with π. Then* $\dim_A(\mathcal{A} - \mathcal{A}) \geq 1$.

Proof. For contradiction, suppose that $\dim_A(\mathcal{A}-\mathcal{A}) < 1$. That is, suppose that there is $\delta \in (0, 1)$ such that $\dim_A(\mathcal{A} - \mathcal{A}) = 1 - \delta$. Let $\mathbf{z} \in \mathcal{A} - \mathcal{A} \setminus \{0\}$, and note that such a choice is possible by the assumption that $\mathbf{b}_1 \neq \mathbf{b}_2$. For each $n \in \mathbb{N}_0$, define $S_n = \{c^n R_{j\theta}\mathbf{z} \mid j = 0, 1, \dots, n\}$. Let $S = \bigcup_{n=0}^{\infty} S_n$, and note that by Lemma 5.1.1, for each $n \in \mathbb{N}_0$, we have

$$S_n \subseteq S \subseteq \mathcal{A} - \mathcal{A}.$$

By the assumption that $\dim_A(\mathcal{A} - \mathcal{A})$ and the above containment, it follows from (2.3.2) that there exists a constant K such that

$$\mathcal{N}_S(r, \rho) \leq K \left(\frac{r}{\rho}\right)^{1-\delta} \tag{5.1.2}$$

for all $0 < \rho < r < 1$.

To obtain a contradiction, let $\varepsilon \in (0, \frac{\pi}{8})$. For each $n \in \mathbb{N}$, take $E_n = \{i\theta(\bmod 2\pi) \mid i = 0, 1, \dots, n\}$. By Lemma 4.3.3, there is a k such that $\delta E_k < \frac{\varepsilon}{2}$, where δE_k is as given in Definition 4.3.2. Take $r = c^k|\mathbf{z}|$ and $\rho = \varepsilon c^k|\mathbf{z}|$. Consider $\mathrm{B}_r(0) \cap S$. To cover this intersection with ρ-balls, it is necessary to cover S_k with ρ-balls, so suppose that $\{\mathbf{p}_i\}_{i=1}^m$ is a collection of points in $\mathbb{R}^2$ (not necessarily in D_k) such that

$$\mathrm{B}_r(0) \cap S_k \subseteq \bigcup_{i=1}^{m} \mathrm{B}_\rho(\mathbf{p}_i).$$

The points of S_k are contained the circle $C_k = \{\mathbf{x} \mid |\mathbf{x}| = c^k|\mathbf{z}|\}$. By the choice of k, the set $C_k \setminus \bigcup_{i=1}^m \mathrm{B}_\rho(\mathbf{p}_i)$ contains no arc of length greater than $\varepsilon c^k|\mathbf{z}| = \rho$. Thus each ρ-ball in the union is contained in a ball of the form $\mathrm{B}_{4\rho}(\mathbf{q}_i)$, where the points $\mathbf{q}_i$ can

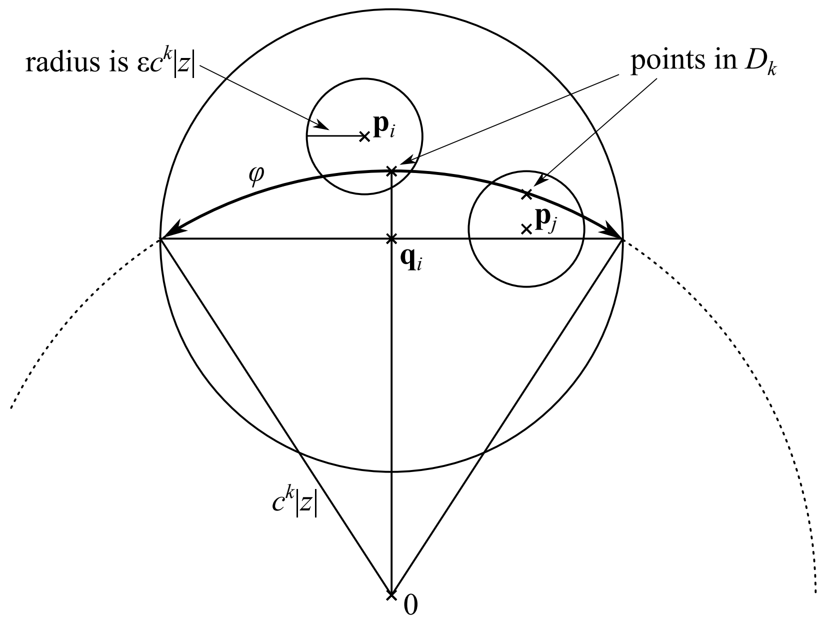

Figure 5.1: Estimating the Number of ρ-balls Covering S_k

be chosen so that a diameter of each ball is a chord of C_k (see Figure 5.1). Moreover, $C_k \subseteq \bigcup_{i=1}^{m} \mathrm{B}_{4\rho}(\mathbf{q}_i)$.

By construction, $C_k \cap \mathrm{B}_{4\rho}(\mathbf{q}_i)$ is an arc of C_k, and all such intersections are congruent. Let φ be the length of each such arc. As $C_k \subseteq \bigcup_{i=1}^{m} \mathrm{B}_{4\rho}(\mathbf{q}_i)$, it follows that $m\varphi \geq 2\pi c^k|\mathbf{z}|$. On the other hand, we have

$$\varphi = 2c^k|\mathbf{z}| \arcsin\left(\frac{4\rho}{r}\right) = 2c^k|\mathbf{z}| \arcsin(4\varepsilon)$$

(again, refer to Figure 5.1). For any $\psi \in (0, \frac{\pi}{2})$, it holds that $\arcsin(\psi) < \frac{\pi}{2}\psi$. Thus $\varphi < 4c^k|\mathbf{z}|\pi\varepsilon$, and so

$$m > \frac{2\pi c^k|\mathbf{z}|}{4c^k|\mathbf{z}|\pi\varepsilon} = \frac{1}{2\varepsilon}.$$

Hence m, the number of ρ-balls required to cover $\mathrm{B}_r(0) \cap S$, is bounded below by

$1/2\varepsilon$. That is,

$$\frac{1}{2\varepsilon} \leq m \leq \mathcal{N}_S(c^k|\mathbf{z}|, \varepsilon c^k|\mathbf{z}|). \tag{5.1.3}$$

Combining the inequalities at (5.1.2) and (5.1.3), we have

$$\frac{1}{2\varepsilon} \leq K\left(\frac{1}{\varepsilon}\right)^{1-\delta},$$

which holds for all $\varepsilon \in (0, \frac{\pi}{8})$. This is a contradiction, as it implies that there is a constant $K^\star = 2K^{1/\delta}$ such that for all ε, it holds that

$$\frac{1}{\varepsilon} \leq K^\star.$$

But $1/\varepsilon$ increases without bound as ε tends to zero, and so no such $K^\star$ can exist. The contradiction is to the assumption that $\dim_A(\mathcal{A} - \mathcal{A}) < 1$, therefore $\dim_A(\mathcal{A} - \mathcal{A}) \geq 1$. $\square$

This result demonstrates that sets in $\mathbb{R}^2$ with arbitrarily small Assouad dimension can have sets of differences with Assouad dimension at least one, which implies that there is no bound on $\dim_A(\mathcal{A} - \mathcal{A})$ in terms of $\dim_A(\mathcal{A})$ alone. Morever, this result can be generalized, as outlined in the following section.

5.2 A Generalization to $\mathbb{R}^D$

The general technique outlined in Section 5.1 can be extended to higher dimensions. Suppose that $O_0 = I, O_1, \ldots, O_{D-1} \in O(D)$ (i.e. O_i is an orthogonal matrix in $\mathbb{R}^{D\times D}$ for each i) are mutually incommensurable in the following sense: For all $\mathbf{x} \in \mathbb{R}^D$ and each $i, j = 0, 1, \ldots, n-1$ with $i \neq j$, we have $\{O_i^p\mathbf{x} \mid p \in \mathbb{N}\} \cap \{O_j^q\mathbf{x} \mid q \in \mathbb{N}\} = \emptyset$.

Now consider the attractor $\mathcal{A}$ of a system of similarities in $\mathbb{R}^D$ given by $F = \{f_i\}_{i=1}^{D}$, with maps of the form

$$f_i(\mathbf{x}) = cO_i\mathbf{x} + \mathbf{b}_i,$$

where $c \in (0, 1)$ is a fixed contraction ratio, the translations $\mathbf{b}_i$ are distinct and chosen so that F satisfies the open set condition, and the O_i are mutually incommensurable as above. Then the set

$$\bigcup_{k=0}^{\infty}\{c^k O^\alpha \mathbf{z} \mid \alpha \in \mathcal{S}_D, \ell(\alpha) = k, \mathbf{z} \in \mathcal{A} - \mathcal{A}\} \subseteq \mathcal{A} - \mathcal{A},$$

where for any $\alpha \in \mathcal{S}_D$, we define $O^\alpha = O_{\alpha_1}O_{\alpha_2}\cdots O_{\alpha_{\ell(\alpha)}}$. At smaller scales, this set is increasingly dense (in the sense of Lemma 4.3.3) in spheres of radius c^k centered at the origin. Thus by assuming the mutual incommensurability of the O_i and the existence of at least one nonzero point $\mathbf{z} \in \mathcal{A} - \mathcal{A}$, we have $\dim_A(\mathcal{A} - \mathcal{A}) \geq D - 1$, which is independent of the choice of c. The existence of a nonzero point is guaranteed by the choice of distinct translations, and examples of the orthogonal matrices of the type required can be easily constructed. For instance, consider the matrices

$$O_0 = I_D, \qquad \text{and} \qquad O_i = \begin{bmatrix} I_i & 0 & 0 \\ 0 & R_\theta & 0 \\ 0 & 0 & I_{D-i-1} \end{bmatrix},$$

where the O_i are block-diagonal for $i \neq 0$. The blocks are a 2×2 rotation matrix R_θ, where $\theta = \eta\pi$ for some $\eta \in \mathbb{R} \setminus \mathbb{Q}$, and $k \times k$ identity matrices I_k, where I_0 is taken to represent an empty block.

Under the hypothesis that F is a system of similarities satisfying the open set condition, the Assouad dimension can be computed using Corollary 3.3.5, and depends on the contraction ratio c and the ambient dimension D. By decreasing the contraction ratio, it is possible to construct systems of the above form with attractors $\mathcal{A}$ of arbitrarily small Assouad dimension.

Thus the techniques used to construct an iterated function system with attractor $\mathcal{A}$ of arbitrarily small Assouad dimension with $\dim_A(\mathcal{A} - \mathcal{A}) \geq 1$ can be generalized to construct systems in $\mathbb{R}^D$ with $\dim_A(\mathcal{A})$ arbitrarily small and $\dim_A(\mathcal{A} - \mathcal{A}) \geq D - 1$ for any $D \in \mathbb{N}$.

5.3 Middle-λ Cantor Sets

Middle-λ Cantor sets are a family of fractal sets that are easily amenable to analysis. The usual construction is as follows: fix some $\lambda \in (0, 1)$ and take C_0 to be the closed unit interval $[0, 1]$. Then for any $k \in \mathbb{N}$, the set C_{k+1} is constructed by removing an open interval of proportional length λ from each of the closed intervals that make up C_k. The middle-λ Cantor set $\mathcal{C}_\lambda$ is the intersection $\bigcap_{k=0}^{\infty} C_k$ (see, for instance, [7, page 14]).

The canonical example of such a set is the middle-third Cantor set (or triadic Cantor dust). The first several iterates of its construction are shown in Figure 5.2. To construct C_1, the open interval $(\frac{1}{3}, \frac{2}{3})$ is removed from $C_0 = [0, 1]$. Thus $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. To construct C_2, the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ are removed from C_1. Thus $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. The middle-third Cantor set is the intersection of the C_i, that is $\bigcap_{i=1}^{\infty} C_i$.

A middle-λ Cantor set may also be represented as the attractor of an iterated

Figure 5.2: The Middle-Third Cantor Set

function system. For any $\lambda \in (0,1)$, the middle-λ Cantor set, denoted $\mathcal{C}_\lambda$, is the attractor of the iterated function system $F_\lambda = \{f_1, f_2\}$ where $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = cx, \qquad \text{and} \qquad f_2(x) = cx + (1-c),$$

where $c = (1-\lambda)/2$. Note that $U = (0,1)$ is a Moran open set for of the system. As the attractor is self-similar, it follows from Corollary 3.3.5 that $\dim_A(\mathcal{C}_\lambda) = \dim_s(\mathcal{C}_\lambda) = \log(2)/\log(\frac{1}{c})$.

Proposition 5.3.1. *The Assouad dimension of the set of differences of the middle-λ Cantor set $\mathcal{C}_\lambda$ is*

$$\dim_A(\mathcal{C}_\lambda - \mathcal{C}_\lambda) = \begin{cases} \dfrac{\log(3)}{\log(\frac{1}{c})} & \textit{if } \lambda \geq \frac{1}{3}; \\[2ex] 1 & \textit{otherwise}, \end{cases}$$

where $c = (1-\lambda)/2$.

Proof. Fix some $\lambda \in (0,1)$ and take $c = (1-\lambda)/2$, as above. Let $\mathcal{B}$ be the attractor of the iterated function system $G = \{g_1, g_2, g_3\}$ given by

$$g_1(z) = cz, \qquad g_2(z) = cz + (1-c) \qquad \text{and} \qquad g_3(z) = cz - (1-c).$$

We claim that $\mathcal{B} = \mathcal{C}_\lambda - \mathcal{C}_\lambda$. Suppose that $z \in \mathcal{B}$. Then there is some sequence

$\gamma = (\gamma_i)_{i=0}^{\infty}$ with $\gamma_i \in \{0, c-1, 1-c\}$ for each i such that

$$z = \sum_{i=0}^{\infty} c^i \gamma_i.$$

Define the sequences $\alpha = (\alpha_i)_{i=0}^{\infty}$ and $\beta = (\beta_i)_{i=0}^{\infty}$ as follows:

$$\begin{cases} \alpha_i = 0, \beta_i = 1-c, & \text{if } \gamma_i = c-1; \\ \alpha_i = 0, \beta_i = 0, & \text{if } \gamma_i = 0; \text{ and} \\ \alpha_i = 1-c, \beta_i = 0, & \text{if } \gamma_i = 1-c. \end{cases}$$

Let

$$x = \sum_{i=0}^{\infty} c^i \alpha_i \qquad \text{and} \qquad y = \sum_{i=0}^{\infty} c^i \beta_i,$$

and note that $x, y \in \mathcal{C}_\lambda$. Then

$$z = \sum_{i=0}^{\infty} c^i \gamma_i = \sum_{i=0}^{\infty} c^i (\alpha_i - \beta_i) = \sum_{i=0}^{\infty} c^i \alpha_i - \sum_{i=0}^{\infty} c^i \beta_i = x - y \in \mathcal{C}_\lambda - \mathcal{C}_\lambda.$$

Hence $z \in \mathcal{C}_\lambda - \mathcal{C}_\lambda$, demonstrating that $\mathcal{B} \subseteq \mathcal{C}_\lambda - \mathcal{C}_\lambda$. The opposite containment is obtained by reversing the above construction, hence the two sets are equal, as claimed.

If $1/3 < \lambda \leq 1$, then G satisfies the open set condition with the open set $V = (-1, 1)$. From this, it immediately follows that

$$\dim_A(\mathcal{C}_\lambda - \mathcal{C}_\lambda) = \dim_A(\mathcal{B}) = \dim_s(\mathcal{B}) = \frac{\log(3)}{\log(\frac{1}{c})}.$$

Otherwise, if $0 < \lambda < 1/3$, then the entire interval $(-1, 1)$ is fixed by G. Thus

$$(-1, 1) \subseteq \mathcal{B} = \mathcal{C}_\lambda - \mathcal{C}_\lambda \subseteq \mathbb{R},$$

hence

$$1 = \dim_A((-1,1)) \leq \dim_A(\mathcal{B}) = \dim_A(\mathcal{C}_\lambda - \mathcal{C}_\lambda) \leq \dim_A(\mathbb{R}) = 1.$$

In either case, the desired result holds. $\square$

Note that for any $\lambda \in (0,1)$, the dimension of $\mathcal{C}_\lambda - \mathcal{C}_\lambda$ is less than twice the dimension of $\mathcal{C}_\lambda$. In this sense, the set of differences of a middle-λ Cantor set is well behaved with respect to the Assouad dimension.

As a specific example, consider the middle-$\frac{3}{5}$ Cantor set. The Assouad dimension of this set is

$$\dim_A(\mathcal{C}_{3/5}) = \frac{\log(2)}{\log\left(\frac{1-\frac{3}{5}}{2}\right)} = \frac{\log(2)}{\log(5)}.$$

Hence $2\dim_A(\mathcal{C}_{3/5}) = \log(4)/\log(5)$. Clearly, $1/3 < 3/5 < 1$, and so by the previous result,

$$\dim_A(\mathcal{C}_{3/5} - \mathcal{C}_{3/5}) = \frac{\log(3)}{\log\left(\frac{1-\frac{3}{5}}{2}\right)} = \frac{\log(3)}{\log(5)}$$

Thus

$$\dim_A(\mathcal{C}_{3/5} - \mathcal{C}_{3/5}) = \frac{\log(3)}{\log(5)} < \frac{\log(4)}{\log(5)} = 2\dim_A(\mathcal{C}_{3/5}).$$

5.4 Asymmetric Cantor Sets

Middle-λ Cantor sets may be formed via an iterative process in which the sets at one stage are formed by removing an open interval of proportional length λ from the center of every subinterval of the previous previous. This process is symmetric in that the intervals that remain at each stage are of equal length. An asymmetric Cantor set may be constructed in a similar fashion, but the intervals are removed from a fixed position other than the center.

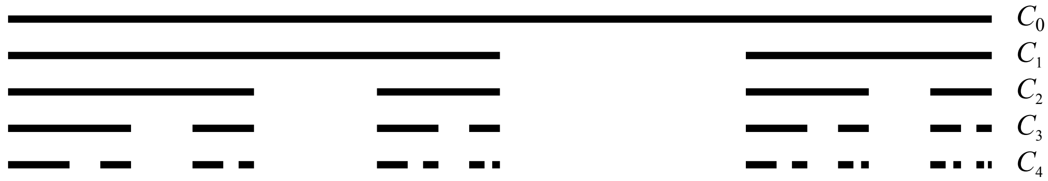

Figure 5.3: Construction of an Asymmetric Cantor Set

For example, in Figure 5.3 the set C_1 is formed by removing an open interval of length $\frac{1}{4}$ from the unit interval so that the remaining intervals are of length $\frac{1}{2}$ and $\frac{1}{4}$. That is, $C_1 = [0, \frac{1}{2}] \cup [\frac{3}{4}, 1]$. The set C_2 is then formed by removing an interval of proportional length $\frac{1}{4}$ from each of the intervals in C_1 so that the interval to the left of a removed interval is twice as long as the interval to the right. Hence $C_2 = [0, \frac{1}{4}] \cup [\frac{3}{8}, \frac{1}{2}] \cup [\frac{3}{4}, \frac{7}{8}] \cup [\frac{15}{16}, 1]$. The intersection of the C_i is an asymmetric Cantor set.

An asymmetric Cantor set is more easily discussed as the attractor of an iterated function system of the form $F_{c_1,c_2} = \{f_1, f_2\}$, where $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = c_1 x, \qquad \text{and} \qquad f_2(x) = c_2 x + (1 - c_2),$$

where $c_1, c_2 \in (0, 1)$. The sets in Figure 5.3 converge to $\mathcal{C}_{\frac{1}{2},\frac{1}{4}}$, where $\mathcal{A}_{c_1,c_2}$ denotes the attractor of F_{c_1,c_2}. In the following, we assume that $c_1 + c_2 < 1$ as, if not, then the entire unit interval is invariant under F_{c_1,c_2}, and so the Assouad dimension of the attractor is one, which is maximal for a subset of $\mathbb{R}$.

The main result of this section is that for any $\varepsilon > 0$, there exist asymmetric Cantor sets $\mathcal{A}_{c_1,c_2}$ with $\dim_A(\mathcal{A}_{c_1,c_2}) < \varepsilon$ and $\dim_A(\mathcal{A}_{c_1,c_2} - \mathcal{A}_{c_1,c_2}) = 1$. That is, there exist sets in $\mathbb{R}$ of arbitrarily small Assouad dimension with sets of differences attaining maximal Assouad dimension. This result is proved in Corollary 5.4.8, however several preliminaries are required.

The proof of Lemma 5.4.6 relies on a number theoretical result pertaining to the approximability of irrational numbers by rational numbers. As the proof is both simple and enlightening, we present it here for completeness.

Theorem 5.4.1 ([10, Theorem 198]). *If $\sum \frac{1}{\chi(q)}$ is convergent, then the set of ξ which satisfy*

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{q\chi(q)}, \tag{5.4.1}$$

for an infinity of $q \in \mathbb{N}$ is of Lebesgue measure zero.

Proof. We may assume without loss of generality that $\xi \in [0,1]$. The real line is contained in a countable union of intervals of unit length, thus if the set of $\xi \in [0,1]$ satisfying (5.4.1) for an infinite number of q is of measure zero, then the set of such ξ in $\mathbb{R}$ is of measure zero.

For every $n \in \mathbb{N}$, consider the collection of intervals of the form

$$I_{np} = \left[\frac{p}{q_n} - \frac{1}{q_n\chi(q_n)}, \frac{p}{q_n} + \frac{1}{q_n\chi(q_n)}\right],$$

where $q_n \geq n$, and $p = 1, 2, \ldots, q_n - 1$. Also define

$$I_{n0} = \left[0, \frac{1}{q_n\chi(q_n)}\right] \qquad \text{and} \qquad I_{n1} = \left[1 - \frac{1}{q_n\chi(q_n)}, 1\right].$$

Let I_n denote the union

$$I_n = \bigcup_{p=0}^{q_n} I_{np}.$$

If ξ satisfies (5.4.1) for infinitely many q, then $\xi \in I_n$ for an infinite number of n. In particular, for any $N \in \mathbb{N}$, there exists at least one $n \geq N$ such that $\xi \in I_n$. The total length of all of such intervals containing ξ, even without eliminating overlaps,

is less than
$$\sum_{n=N}^{\infty}\sum_{p=0}^{q_n}\frac{2}{q_n\chi(q_n)}=\sum_{n=N}^{\infty}\frac{2q_n}{q_n\chi(q_n)}=2\sum_{n=N}^{\infty}\frac{1}{\chi(q_n)},$$
which converges to zero as $N\to\infty$ by hypothesis. Hence any ξ satisfying (5.4.1) is contained in a set of arbitrarily small Lebesgue measure. □

This theorem shows that for almost every $\xi\in\mathbb{R}$, for any $\varepsilon>0$ and any constant C, there are only finitely many rational approximations of ξ which satisfy
$$\left|\frac{p}{q}-\xi\right|<\frac{C}{q^{2+\varepsilon}}. \tag{5.4.2}$$
If $\xi\in\mathbb{R}\setminus\mathbb{Q}$ is such that there are infinitely many q satisfying (5.4.2) for all $\varepsilon>0$, then we say that ξ is *well approximable by rationals*. Otherwise, we say that ξ is *badly approximable by rationals*. This result and terminology is discussed in greater detail in several elementary texts. See, for instance, Burger [4, Module 9] or Sally and Sally [25, Chapter 4].

Throughout the following lemmas, we assume that $c_1,c_2\in(0,1)$ are fixed, and we simplify the notation by omitting the subscripts. That is, we let F denote the system F_{c_1,c_2} and $\mathcal{A}$ denote the attractor of F. For brevity, let $\mathbb{N}_0$ be the set of nonnegative integers, i.e. $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Lemma 5.4.2. *Define*
$$D=\{c_1^pc_2^q\mid p,q\in\mathbb{N}_0\}. \tag{5.4.3}$$
Then $D\subseteq\mathcal{A}-\mathcal{A}$.

Proof. Note that $f_1(0)=0$ and that $f_2(1)=1$. Thus both 0 and 1 are fixed by the system F, and so $0,1\in\mathcal{A}$. As 0 and 1 are in the attractor, so to are their images

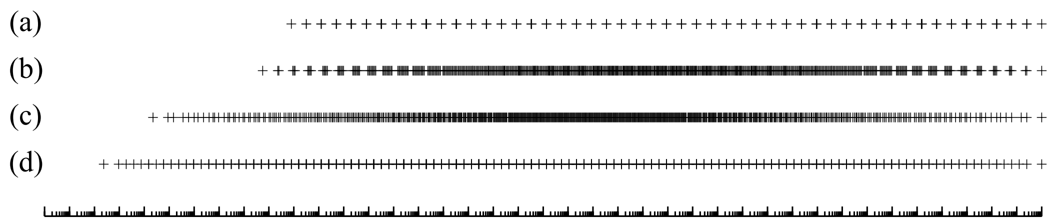

Figure 5.4: Log-scale Graphs of D_{25} ($c_1 = \frac{1}{4}$, c_2 Varies)

under any sequence of maps in F. That is, $f^\alpha(0), f^\beta(1) \in \mathcal{A}$ for all $\alpha, \beta \in \mathcal{S}_2$. Let $p, q \in \mathbb{N}_0$. Clearly $f_1^p(x) = c_1^p x$. Note that $f_2(0) = 1 - c_2$, and assume for induction that $f_2^{q-1}(0) = 1 - c_2^{q-1}$. Then

$$f_2^q(0) = f_2 \circ f_2^{q-1}(0) = f_2(1 - c_2^{q-1}) = c_2 - c_2^q + 1 - c_2 = 1 - c_2^q,$$

and so $f_2^q(0) = 1 - c_2^q$ for all $q \in \mathbb{N}_0$. Thus

$$c_1^p c_2^q = c_1^p - (c_1^p - c_1^p c_2^q) = f_1^p(1) - f_1^p(1 - c_2^q) = f_1^p(1) - (f_1^p \circ f_2^q)(0) \in \mathcal{A}.$$

The result holds for all $p, q \in \mathbb{N}_0$, and so $D \subseteq \mathcal{A} - \mathcal{A}$. $\square$

Lemma 5.4.2 isolates a countable subset D of $\mathcal{A} - \mathcal{A}$. We will show that this countable subset is of maximal Assouad dimension for almost all choices of contraction ratios c_1 and c_2. Figure 5.4 shows, with log scaling, the sets $D_{25} = \{c_1^p c_2^q \mid 0 \leq p, q \leq 25\}$, where $c_1 = 1/4$ and (a) $c_2 = 1/4$, (b) $c_2 = 9/40$, (c) $c_2 = 3/20$, and (d) $c_2 = 1/8$. The points in (a) and (d) fall onto a logarithmic grid. These sets, as well as the attractors that contain them, may have Assouad dimension strictly less than 1, as is shown in Proposition 5.4.9.

Figure 5.4 (b) and (c) are more interesting. As noted in Section 2.3, the Assouad dimension is sensitive to local complexity, and the dimension of a set can often be determined by a countable subset. In (b) and (c), the points shown seem to "fill up" a portion of the unit interval. This is a qualitative indication that these sets may display the kind of local complexity to which the Assouad dimension is sensitive. This result is made formal in Proposition 5.4.7.

Lemma 5.4.3. *For each $n \in \mathbb{N}$, let $\mathcal{I}_n$ denote the set of indices*

$$\mathcal{I}_n = \{(p,q) \in \mathbb{N}_0^2 \mid c_2^n < c_1^p c_2^q \leq c_2^{n-1}\}. \tag{5.4.4}$$

Then

$$\operatorname{card}(\mathcal{I}_n) = n \left\lfloor \frac{\log(c_2)}{\log(c_1)} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Proof. Fix some $n \in \mathbb{N}$. Suppose that $p \in \mathbb{N}_0$ with $p \leq n \log(c_2)/\log(c_1)$. Then

$$n - p\frac{\log(c_1)}{\log(c_2)} \geq 0.$$

There is a unique $q \in \mathbb{N}_0$ such that

$$n - p\frac{\log(c_1)}{\log(c_2)} > q \geq n - p\frac{\log(c_1)}{\log(c_2)} - 1. \tag{5.4.5}$$

Note that if p is taken to be larger than $n\log(c_2)/\log(c_1)$, then there is no nonnegative q satisfying (5.4.5). As $c_2 < 1$, $\log(c_2) < 0$, so we multiply through by $\log(c_2)$ to obtain

$$n\log(c_2) - p\log(c_1) < q\log(c_2) \leq (n-1)\log(c_2) - p\log(c_1)$$

Further manipulation of the inequalities renders

$$n \log(c_2) < p \log(c_1) + q \log(c_2) \leq (n-1) \log(c_2),$$

which implies

$$c_2^n < c_1^p c_2^q \leq c_2^{n-1}.$$

Thus for each p with $0 \leq p \leq n \log(c_2)/\log(c_1)$, there exists a unique q such that $(p, q) \in \mathcal{I}_n$. There are $n\lfloor \log(c_2)/\log(c_1) \rfloor + 1$ such p, and the inequality at (5.4.5) is uniquely satisfied by some q for each p. Thus $\operatorname{card}(\mathcal{I}_n) = \lfloor \log(c_2)/\log(c_1) \rfloor + 1$. □

Having established the number of points in D contained in each interval of the form $[c_2^n, c_2^{n-1}]$, we now show that, for most choices of c_1 and c_2, no two points are "too close together." This is done in Lemma 5.4.6, following two technical lemmas.

Lemma 5.4.4. *Let $\mathcal{I}_n$ be as in Lemma 5.4.3, and let $(p_1, q_1), (p_2, q_2) \in \mathcal{I}_n$. Then $p_1 < p_2$ implies that $q_2 \leq q_1$.*

Proof. There are two cases to consider: either $c_1^{p_1} c_2^{q_1} \leq c_1^{p_2} c_2^{q_2}$ or $c_1^{p_1} c_2^{q_1} > c_1^{p_2} c_2^{q_2}$.

(1) Suppose that $c_1^{p_1} c_2^{q_1} \leq c_1^{p_2} c_2^{q_2}$. As $p_1 < p_2$, it follows that $c_1^{p_2} < c_1^{p_1}$. Multiplying by $c_2^{q_2}$, we obtain $c_1^{p_2} c_2^{q_2} < c_1^{p_1} c_2^{q_2}$. Thus $c_1^{p_1} c_2^{q_1} < c_1^{p_1} c_2^{q_2}$. Canceling $c_1^{p_1}$, we have $c_2^{q_1} < c_2^{q_2}$, which implies that $q_2 < q_1$.

(2) Suppose that $c_1^{p_1} c_2^{q_1} > c_1^{p_2} c_2^{q_2}$, and assume for contradiction that $q_1 < q_2$. As above, $c_1^{p_2} < c_1^{p_1}$, and so $c_2^{p_2} c_2^{q_2} < c_1^{p_1} c_2^{q_2}$. But then, as $c_2^{q_2} < c_2^{q_1}$, we have

$$c_2^n < c_1^{p_2} c_2^{q_2} < c_1^{p_1} c_2^{q_2} < c_1^{p_1} c_2^{q_1} \leq c_2^{n-1}.$$

Hence $(p_1, q_2) \in \mathcal{I}_n$. However, as noted in (5.4.5), for each p, there is a unique q

such that $(p, q) \in \mathcal{I}_n$. This implies that $q_1 = q_2$, which is a contradiction. Thus $q_2 \le q_1$.

In either case, the desired inequality holds, thus completing the proof. □

Lemma 5.4.5. *Assume that* $c_1 > c_2$. *Let* $\mathcal{I}_n$ *be as in Lemma 5.4.3, and let* $(p_1, q_1), (p_2, q_2) \in \mathcal{I}_n$ *with* $(p_1, q_1) \neq (p_2, q_2)$. *Define*

$$m_n = \min\left\{ \left|1 - \frac{c_1^p}{c_2^q}\right| \;\middle|\; 0 \le p, q \le n\frac{\log(c_2)}{\log(c_1)} \right\}. \tag{5.4.6}$$

Then $|c_1^{p_1} c_2^{q_1} - c_1^{p_2} c_2^{q_2}| \ge c_2^n m_n$.

Proof. Without loss of generality, assume that $p_1 < p_2$. Then by Lemma 5.4.4, $q_2 \le q_1$. Thus $p_2 - p_1 > 0$ and $q_2 - q_1 \ge 0$. Manipulating $|c_1^{p_1} c_2^{q_1} - c_1^{p_2} c_2^{q_2}|$, we obtain

$$|c_1^{p_1} c_2^{q_1} - c_1^{p_2} c_2^{q_2}| = c_1^{p_1} c_2^{q_1} \left|1 - \frac{c_1^{p_2 - p_1}}{c_2^{q_1 - q_2}}\right| > c_2^n \left|1 - \frac{c_1^{p_2 - p_1}}{c_2^{q_1 - q_2}}\right|. \tag{5.4.7}$$

By the proof of Lemma 5.4.3, $p_2 - p_1 < p_2 \le n \log(c_2)/\log(c_1)$. From this, it follows that

$$p_2 - p_1 \le n\frac{\log(c_2)}{\log(c_1)}.$$

From the construction of $\mathcal{I}_n$ (see (5.4.4), it is clear that $q_1 - q_2 \le q_1 < n$. As $c_1 > c_2$, we have $\log(c_1) > \log(c_2)$, which implies that $1 < \log(c_2)/\log(c_1)$. Thus

$$q_1 - q_2 \le q_1 < n < n\frac{\log(c_2)}{\log(c_1)}.$$

Then, taking minimums at (5.4.7), we have

$$|c_1^{p_1} c_2^{q_1} - c_1^{p_2} c_2^{q_2}| \ge c_2^n \min\left\{ \left|1 - \frac{c_1^p}{c_2^q}\right| \;\middle|\; 0 < p, q \le n\frac{\log(c_2)}{\log(c_1)} \right\} = c_2^n m_n,$$

which is the desired result. □

Lemma 5.4.6. *Suppose $c_1, c_2 \in (0,1)$ with $c_1 > c_2$ are such that $\log(c_2)/\log(c_1)$ is badly approximable by rationals. For each $n \in \mathbb{N}$, let m_n be as in Lemma 5.4.5. Then for any $\delta > 0$, there is a constant $C_{n\delta}$ such that*

$$m_n \geq C_{n\delta} \frac{1}{n^{1+\delta}}.$$

Proof. The proof is by contradiction. Fix $\delta > 0$ and suppose that for every $k \in \mathbb{N}$, there exists some n_k such that $m_{n_k} < (1/k)/(n_k^{1+\delta})$. For each such n_k, let p_{n_k} and q_{n_k} be such that $m_n = \left|1 - \frac{c_1^{p_{n_k}}}{c_2^{q_{n_k}}}\right|$. Then

$$-\frac{1}{n_k^{1+\delta}k} < 1 - \frac{c_1^{p_{n_k}}}{c_2^{q_{n_k}}} < \frac{1}{n_k^{1+\delta}k}, \qquad \text{thus} \qquad 1 + \frac{1}{n_k^{1+\delta}k} > \frac{c_1^{p_{n_k}}}{c_2^{q_{n_k}}} > 1 - \frac{1}{n_k^{1+\delta}k},$$

and so

$$\log\left(1 + \frac{1}{n_k^{1+\delta}k}\right) > p_{n_k}\log(c_1) - q_{n_k}\log(c_2) > \log\left(1 - \frac{1}{n_k^{1+\delta}k}\right). \tag{5.4.8}$$

If $x \in (0,1)$, then $\log(1+x) \leq x$. Moreover, for any $\eta \in (-1, -x)$, it follows from the concavity of the logarithm function that $-\frac{1}{\eta}\log(1+\eta)x < \log(1-x)$. But $\frac{1}{\eta}\log(1+\eta) > 1$, and so $\frac{1}{\eta}\log(1+\eta)x > x$. This implies that for any $x \in (0,1)$, there exists a constant c such that $-cx < \log(1-x) < \log(1+x) < cx$. In particular, note that $1/(n_k^{1+\delta}k) \in (0,1)$, and so there is a constant c such that

$$-c\frac{1}{n_k^{1+\delta}k} < \log\left(1 - \frac{1}{n_k^{1+\delta}k}\right) < \log\left(1 + \frac{1}{n_k^{1+\delta}k}\right) c\frac{1}{n_k^{1+\delta}k}.$$

Hence the inequality at (5.4.8) becomes

$$\frac{c}{n_k^{1+\delta}k} > p_{n_k}\log(c_1) - q_{n_k}\log(c_2) > -\frac{c}{n_k^{1+\delta}k}$$
$$\frac{c}{n_k^{1+\delta}k} + q_{n_k}\log(c_2) > p_{n_k}\log(c_1) > -\frac{c}{n_k^{1+\delta}k} + q_{n_k}\log(c_2)$$
$$-\frac{c}{q_{n_k}n_k^{1+\delta}k\log(\frac{1}{c_1})} + \frac{\log(\frac{1}{c_2})}{\log(\frac{1}{c_1})} < \frac{p_{n_k}}{q_{n_k}} < \frac{c}{q_{n_k}n_k^{1+\delta}k\log(c_1)} + \frac{\log(\frac{1}{c_2})}{\log(\frac{1}{c_1})}$$
$$\left|\frac{p_{n_k}}{q_{n_k}} - \frac{\log(\frac{1}{c_2})}{\log(\frac{1}{c_1})}\right| < \frac{c}{q_{n_k}n_k^{1+\delta}(\log(\frac{1}{c_1}))}. \tag{5.4.9}$$

We have $q_{n_k} < n_k\log(c_2)/\log(c_1)$ from the construction of m_{n_K}. Applying this to (5.4.9), we have

$$\left|\frac{p_{n_k}}{q_{n_k}} - \frac{\log(\frac{1}{c_2})}{\log(\frac{1}{c_1})}\right| < \left(\frac{-c\log(\frac{1}{c_2})^{1+\delta}}{\log(\frac{1}{c_1})^{2+\delta}}\right)\frac{1}{q_{n_k}^{2+\delta}}.$$

The term in parentheses is a constant, and p_{n_k}/q_{n_k} is rational. Hence there are infinitely many q satisfying (5.4.2). This contradicts the hypothesis that $\log(c_2)/\log(c_1)$ is badly approximable by rationals. Therefore for every $n \in \mathbb{N}$ and any choice of $\delta > 0$, there exists a constant $C_{n\delta}$ such that

$$m_n \geq C_{n\delta}\frac{1}{n^{1+\delta}}.$$

$\square$

Proposition 5.4.7. *Suppose $c_1, c_2 \in (0,1)$ are such that $\log(c_2)/\log(c_1)$ is badly approximable by rationals, and let $\mathcal{A}$ be the attractor of the system $F = \{f_1, f_2\}$,* *given by*

$$f_1(x) = c_1 x, \qquad \textit{and} \qquad f_2(x) = c_2 x + (1 - c_2).$$

Then $\dim_A(\mathcal{A} - \mathcal{A}) = 1$*.*

Proof. We may assume without loss of generality that $c_1 > c_2$. If $c_1 + c_2 \geq 1$, then the unit interval is fixed by F, and so $[-1, 1] \subseteq \mathcal{A} - \mathcal{A} \subseteq \mathbb{R}$. This implies that

$$1 = \dim_A([-1, 1]) \leq \dim_A(\mathcal{A} - \mathcal{A}) \leq \dim_A(\mathbb{R}) = 1,$$

and so $\dim_A(\mathcal{A} - \mathcal{A}) = 1$.

Now suppose that $c_1 + c_2 < 1$, and for contradiction, assume that $\dim_A(\mathcal{A} - \mathcal{A}) = a < 1$. It follows from (2.3.2) that there is some constant K such that

$$\mathcal{N}_{\mathcal{A}-\mathcal{A}}(r, \rho) \leq K \left(\frac{r}{\rho}\right)^a$$

for all $0 < \rho < r < 1$. For each $n \in \mathbb{N}$, let

$$r_n = \frac{1}{2} c_2^{n-1}(1 - c_2) \qquad \text{and} \qquad \rho_n = c_2^n \frac{1}{n^{1+\delta}},$$

where $0 < \delta < \frac{1}{a} + 1$ is chosen so that $a(1 + \delta) < 1$. Note that this is possible, as $a < 1$. Then

$$\mathcal{N}_{\mathcal{A}-\mathcal{A}}(r_n, \rho_n) \leq K \left(\frac{r_n}{\rho_n}\right)^a = K \left(\frac{(1 - c_2)}{c_2} n^{1+\delta}\right)^a. \qquad (5.4.10)$$

To estimate $\mathcal{N}_{\mathcal{A}-\mathcal{A}}(r_n, \rho_n)$ from below, let D be as in Lemma 5.4.2, and note that by that lemma, $D \subseteq \mathcal{A} - \mathcal{A}$. Let I_n denote the interval $[c_2^n, c_2^{n-1}]$, a ball of radius r_n. Let m_n be as in Lemma 5.4.5, and note that as $\log(c_2)/\log(c_1)$ is badly approximable, there is some C such that

$$c_2^n m_n \geq c_2^n C \frac{1}{n^{1+\delta}} > \rho_n.$$

Thus for any two $x, y \in D$, it follows from Lemma 5.4.5 that $|x - y| \geq c_2^n m_n > \rho_n$.

As the distance between any two points in $I_n \cap D$ is greater than ρ_n, no two points in $I_n \cap D$ can be covered by a single ball of radius ρ_n. By Lemma 5.4.3, $I_n \cap D$ contains at least $n \log(c_2)/\log(c_1)$ points, thus

$$n \frac{\log(c_2)}{\log(c_1)} \leq \mathcal{N}_{\mathcal{A}-\mathcal{A}}(r_n, \rho_n). \tag{5.4.11}$$

Combining the inequalities at (5.4.10) and (5.4.11), we obtain

$$n \frac{\log(c_2)}{\log(c_1)} \leq \mathcal{N}_{\mathcal{A}-\mathcal{A}}(r_n, \rho_n) \leq K \left(\frac{(1 - c_2)}{2c_2} n^{1+\delta} \right)^a.$$

Combining the constants into a single term

$$K^\star = K \left(\frac{1 - c_2}{2c_2} \right)^a \frac{\log(c_1)}{\log(c_2)},$$

we have $n \leq K^\star n^{a(1+\delta)}$, which implies that

$$n^{1-a(1+\delta)} \leq K^\star$$

for all $n \in \mathbb{N}$. But δ was chosen so that $1 - a(1 + \delta) > 0$, which implies that the term on the left increases without bound as n tends to infinity. Hence no such $K^\star$ can exist, which contradicts the assumption that $\dim_A(\mathcal{A} - \mathcal{A}) = a < 1$. $\square$

An immediate consequence of Proposition 5.4.7 is the following.

Corollary 5.4.8. *For every* $\varepsilon > 0$*, there exists a set* $\mathcal{A} \subseteq \mathbb{R}$ *such that* $\dim_A(\mathcal{A}) < \varepsilon$ *and* $\dim_A(\mathcal{A} - \mathcal{A}) = 1$.

Proof. Fix $\varepsilon > 0$ and choose $c_1, c_2 \in (0,1)$ such that $c_1^\varepsilon + c_2^\varepsilon = 1$ and $\log(c_1)/\log(c_2)$ is badly approximable by rationals. Let $F = \{f_1, f_2\}$ be given by

$$f_1(x) = c_1 x, \qquad \text{and} \qquad f_2(x) = c_2 x + (1 - c_2),$$

and denote the attractor of F by $\mathcal{A}$. F satisfies the open set condition, and so $\dim_A(\mathcal{A}) = \varepsilon$. However, by Proposition 5.4.7, $\dim_A(\mathcal{A} - \mathcal{A}) = 1$. $\square$

Proposition 5.4.7 and Corollary 5.4.8 prove that when $\log(c_2)/\log(c_1)$ is badly approximable by rationals, the asymmetric Cantor set corresponding to the contraction ratios c_1 and c_2 has a set of differences of maximal Assouad dimension. This matches the intuition of Figure 5.4. As noted above, the graphs labeled (b) and (c) correspond to ratios that are badly approximable by rationals, and hence fill up part of the unit interval.

There are at least two other cases to consider: $\log(c_2)/\log(c_1)$ can be either rational, or well approximable by rationals. The case of well approximability is currently an open question, but in the case that $\log(c_2)/\log(c_1)$ is rational, the Assouad dimension can sometimes be bounded. We have already shown in Proposition 5.3.1 that if $\log(c_1)/\log(c_2) = 1$, then the Assouad dimension can be computed exactly. Intuitively, graphs (a) and (d) in Figure 5.4 show points that fall onto a grid, indicating that the underlying set of differences may be well-structured enough to allow computation of the Assouad dimension. This leads to the following proposition.

Proposition 5.4.9. *Suppose that $c \in (0,1)$, that $p, q \in \mathbb{N}$, and let $\mathcal{A}$ be the attractor of the system $F = \{f_1, f_2\}$, given by*

$$f_1(x) = c^p x, \qquad \text{and} \qquad f_2(x) = c^q x + (1 - c^q).$$

Then $\log(3)/(p+q)\log(\frac{1}{c}) \leq \dim_A(\mathcal{A} - \mathcal{A}) \leq \log(3)/\log(\frac{1}{c})$.

Proof. Using techniques similar to those in Proposition 5.3.1, it is possible to show that $\mathcal{C}_{c^{p+q}} \subseteq \mathcal{A} \subseteq \mathcal{C}_c$, where $\mathcal{C}_\lambda$ is the middle-λ Cantor set, as described in Section 5.3. The desired result follows from Proposition 5.3.1. □

5.5 Consequences

Falconer [7] notes that any similarity may be written as the composition of a contraction, a translation, a rotation, and a possible reflection.

In Section 5.1, we used only the properties of rotations to construct self-similar attractors of arbitrarily small Assouad dimension possessing sets of differences of Assouad dimension at least one. This can be generalized to construct self-similar attractors $\mathcal{A} \in \mathbb{R}^D$ with $\dim_A(\mathcal{A}) < \varepsilon$ and $\dim_A(\mathcal{A} - \mathcal{A}) \geq D - 1$, again using only the properties of the rotations involved. In Section 5.4, we constructed asymmetric Cantor sets of arbitrarily small Assouad dimension possessing sets of differences of maximal Assouad dimension using only the properties of varying contraction ratios.

In the full generality of iterated function systems of similarities, the results of these sections can be combined to produce attractors which have arbitrarily small Assouad dimension, but which possess sets of differences of maximal Assouad dimension. Contrary to what might be hoped, this indicates that even in simplified settings, the set of differences can be complicated. Indeed, as the sets of badly approximable numbers and irrational numbers are each of full measure, it is in some sense "normal" for the attractor $\mathcal{A}$ of an iterated function system of similarities in $\mathbb{R}^D$ to have a set of differences $\mathcal{A} - \mathcal{A}$ such that $\dim_A(\mathcal{A} - \mathcal{A}) = D$.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

The Assouad dimension is of practical interest, as it provides a means for quantifying the complexity of sets, particular those that arise as the attractors of both discrete- and continuous-time dynamical systems. With such a quantification, it is possible in principle to demonstrate that a problem posed in a high dimensional space has lower dimensional dynamics and that the system may be addressed computationally in the lower dimensional setting. In particular, if the set of differences of the attractor of a dynamical system is of finite Assouad dimension, then the attractor itself can be embedded into a finite dimensional space without losing the dynamics of the original system. Thus the complexity of a problem may be stated in terms of the Assouad dimension of a set of differences.

In this thesis, we were interested in the question of whether or not it is possible to bound the Assouad dimension of the set of differences in terms of the Assouad dimension of the original set for any class of sets. As noted in Chapter 5, it has been shown in the abstract that there exist sets of arbitrarily small Assouad dimension that have sets of differences of maximal Assouad dimension.

Having built a solid theory of self-similar sets, we were then able to construct iterated function systems in $\mathbb{R}^D$ with attractors of arbitrarily small Assouad dimension possessing sets of differences of maximal Assouad dimension. Intuitively, this result indicates that for the vast majority of sets that occur as the attractors of dynamical systems, it is likely that the set of differences will be of great complexity. This implies that for most such attractors, the results outlined in Section 2.4 cannot be used to obtain lower dimensional embeddings.

6.2 Future Work

Question 6.2.1. Given a fractal set $\mathcal{A}_1$, let $(\mathcal{A}_n)_{n=1}^{\infty}$ be the sequence of sets obtained by setting $\mathcal{A}_{n+1} = \mathcal{A}_n - \mathcal{A}_n$. Suppose that $\dim_A(\mathcal{A}_2) \leq 2\dim_A(\mathcal{A}_1)$. What, if anything, can be said about $\dim_A(\mathcal{A}_n)$?

Question 6.2.2. In several of the examples in Chapter 4, we examined various iterated function systems that were grid like but which did not satisfy the open set condition. In several cases, we were able to compute exact values for the Assouad dimension by constructing new iterated function systems that had the same attractor as the original system, but which satisfied the open set condition. This proves to be a useful technique for computation, but also raises the question of whether or not such a construction is always possible.

More formally, we say that two iterated function systems are *equivalent* if they have the same attractor. Currently, there are no known examples of grid like systems that are not equivalent to some self-similar system which satisfies the open set condition. Is this always the case? or are there examples of grid like systems that are not equivalent to any system satisfying the open set condition? If the latter, under what conditions is a grid like system equivalent to a system satisfying the open set condition?

Conversely, if F is equivalent to a self-similar system G which satisfies the open set condition, is it possible to conclude that F is grid like? What additional hypotheses on G are required in order to guarantee that F is grid like?

Question 6.2.3. Fix a $D \times D$ diagonal matrix C with entries in $(0,1)$, and let $r \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^D$. The set

$$E_r(p) = \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; \left|\left\langle \mathbf{x} - \mathbf{p}, C^{\log(1/r)}(\mathbf{x} - \mathbf{p}) \right\rangle\right| \leq r^2 \right\}$$

is the *C-anisotropic ball of radius r centered at* $\mathbf{p}$. An anisotropic ball of radius 1 is simply a ball of radius 1. For other radii, an anisotropic ball is a ball that has been stretched or flattened in one direction. As the radius of an anisotropic ball decreases, the flattening becomes more profound, which parallels the flattening that occurs to images of sets under self-affine iterated function systems, and the local behavior of the attractors of some partial differential equations.

Define $\mathcal{N}_{\mathcal{A}}^C(r,\rho)$ to be the number of C-anisotropic balls of radius ρ centered in $\mathcal{A}$ required to cover any C-anisotropic ball of radius r centered in $\mathcal{A}$. Then we may take the *C-anisotropic Assouad dimension of* $\mathcal{A}$ to be the infimal number value $a \in \mathbb{R}$ for which there is some constant K such that

$$\mathcal{N}_{\mathcal{A}}^C(r,\rho) \leq K \left(\frac{r}{\rho}\right)^a$$

for all $0 < \rho < r < 1$. We denote the anisotropic Assouad dimension by $\dim_A^C(\mathcal{A}) = a$.

Given these definitions, can the C-anisotropic Assouad dimension of a set be computed for any sets of interest? Can it be computed for sets of differences? Finally, assuming that these computations are possible, can the embedding results of Olson and Robinson [23] be modified to obtain a useful statement for the anisotropic Assouad dimension?

Question 6.2.4. To show that most asymmetric Cantor sets have sets of differences of maximal Assouad dimension, we use the poor approximability of a particular subset of differences. This suggests the following question: do there exist choices $c_1, c_2 \in (0,1)$ such that $\log(c_2)/\log(c_1) \in \mathbb{R} \setminus \mathbb{Q}$ is well approximable by rationals but the attractor $\mathcal{A}$ of the asymmetric Cantor set with contraction ratios c_1 and c_2 is such that $\dim_A(\mathcal{A} - \mathcal{A}) < 1$?

Question 6.2.5. As proved by Luukkainen [16], $\dim_A(\mathcal{A} \times \mathcal{B}) \leq \dim_A(\mathcal{A}) + \dim_A(\mathcal{B})$ for any compact metric spaces $\mathcal{A}$ and $\mathcal{B}$, with equality holding if $\mathcal{A} = \mathcal{B}$. Olson [22, Theorem 3.2] proves a similar result, though as noted by Luukkainen (personal communication), there is an error in the proof of the reverse inequality. Specifically, the first displayed equation on page 466 of the article does not generally imply that $\dim_A(\mathcal{A}) + \dim_A(\mathcal{B}) \leq \dim_A(\mathcal{A} \times \mathcal{B})$. In the case that $\mathcal{A} = \mathcal{B}$, we have $\mathcal{N}_{\mathcal{A}}(r, \rho) \leq K^{1/2}(r/\rho)^{d/2}$ in the final displayed equation on page 465, from which the desired inequality follows. However, in an example due to Larman [14], it is shown that the reverse inequality may be strict, and so the desired result need not hold in general. Rather, $\dim_A(\mathcal{A} \times \mathcal{B}) \leq \dim_A(\mathcal{A}) + \dim_A(\mathcal{B})$ may be strict.

While the above inequality may be strict, it is possible for equality to hold. For instance, as noted above, if $\mathcal{B} = \mathcal{A}$, we have $\dim_A(\mathcal{A} \times \mathcal{A}) = 2\dim_A(\mathcal{A})$. What additional assumptions are sufficient in order for equality to hold? In particular, if

$\mathcal{A}$ and $\mathcal{B}$ have additional structure, such as being the attractors of grid like iterated function systems, is it possible to prove that $\dim_A(\mathcal{A} \times \mathcal{B}) \leq \dim_A(\mathcal{A}) + \dim_A(\mathcal{B})$?

Question 6.2.6. Eden, Kalanarov, and Zelik [5] use techniques from Floquet theory to construct continuous time dynamical systems with smooth attractor which cannot be embedded into any log-Lipschitz manifold. Is it possible to use the techniques of Eden, Kalanarov, and Zelik in order to construct concrete examples of dynamical systems with attractors that behave in a manner similar to that of the attractors of the iterated function systems discussed in Chapter 5?

In particular, let $\mathscr{H}$ be a Hilbert space. Is it possible to construct concrete examples of nonlinear maps on $\mathscr{H}$ possessing relatively simple attractors that cannot be embedded into finite dimensional spaces? Is there an example of a map $F : \mathscr{H} \to \mathscr{H}$ with attractor $\mathcal{A}$ such that $\dim_A(\mathcal{A}) < \varepsilon$ but $\dim_A(\mathcal{A} - \mathcal{A}) = +\infty$?

Question 6.2.7. Several recent works have computed the Assouad dimension of certain self-affine sets in $\mathbb{R}^2$ by considering projections of the sets in one direction and cross-sections in another. For instance, see Mackay [17] and Fraser [9]. More formally, they consider sets that are attractors of systems of the form $F = \{f_i\}_{i=1}^{L}$ with maps given by

$$f_i(x, y) = \begin{bmatrix} c_{1i} & 0 \\ 0 & c_{2i} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_{1i} \\ b_{2i} \end{bmatrix},$$

where the translations and contractions are chosen so that the unit square $[0, 1]^2$ is a Moran open set for the system. Additionally, they constrain the maps in the following manner: let G and H be iterated function system in $\mathbb{R}$ with maps

$$g_i(x) = c_{1i}x + b_{1i}, \qquad \text{and} \qquad h_i(y) = c_{2i}y + b_{2i},$$

so that we may write

$$f_i(x,y) = \begin{bmatrix} g(x) \\ h(y) \end{bmatrix}.$$

The system G, modulo any duplicate maps, must satisfy the open set condition, and the system H must have cross-sections that satisfy the open set condition in the following sense: for each $i = 1, 2, \ldots, L$, let I_i be the collection of indices j such that $g_i = g_j$, and let $H_i = \{h_j \mid j \in I_i\}$. For each i, H_i is an iterated function system. It is required that each such system satisfies the open set condition.

Under these conditions, it is generally possible to compute the Assouad dimension of the attractor $\mathcal{A}$ of F. In principle, we may weaken the hypotheses. For instance, we might require that F satisfy the open set condition, but that G and the H_i are only grid like. Is it still possible to compute the Assouad dimension of F? Assuming this is possible, can we further weaken the hypotheses to require that F be only grid like and still compute the Assouad dimension?

Question 6.2.8. Let $\mathscr{H}$ be a Hilbert space. Hutchinson [13] defines a similarity f on $\mathscr{H}$ as a map of the form

$$f(\mathbf{x}) = cU\mathbf{x} + \mathbf{b},$$

where $c \in (0,1)$ is an arbitrary scaling, U is a unitary transformation on $\mathscr{H}$, and $\mathbf{b} \in \mathscr{H}$ is an arbitrary translation. Hutchinson shows that if $F = \{f_i\}_{i=1}^{L}$ is a finite system of similarities, then F has a unique compact invariant set $\mathcal{A}$.

Now consider the case in which $F = \{f_i\}_{i=1}^{\infty}$, a system consisting of countably many similarities on $\mathscr{H}$. Under what conditions does F have a compact invariant set? If F possesses such an invariant set, under what conditions is it unique?

For instance, consider the following conditions:

(1) Let c_i denote the contraction ratio of the map f_i, and suppose that there is some finite, positive s such that $\sum_{i=1}^{\infty} c_i^s = 1$.

(2) Suppose that F possess a compact absorbing set.

(3) Suppose that F possess a Moran open set.

Are any of these conditions, either alone or in combination, enough to ensure that F has a compact invariant set? If so, will this set be unique?

Additionally, suppose that F does have a unique invariant set $\mathcal{A}$. Can that fractal or Assouad dimension of this set be calculated, and under what conditions will it be finite? For instance, if condition (1) is satisfied, then $\mathcal{A}$ has a well-defined similarity dimension. Under what circumstances will this similarity dimension correspond to the fractal or Assouad dimension? Is the open set condition sufficient for equality to hold? Is there an extension of the grid like concept to a Hilbert space which would ensure a bound of the kind given in Proposition 3.2.3?

Question 6.2.9. In Chapter 5, we showed that the Moran open set condition is not sufficient to ensure that $\dim_A(\mathcal{A}-\mathcal{A}) \leq 2\dim_A(\mathcal{A})$, though there are iterated function systems satisfying the Moran open set condition for which such a bound holds. More generally, there are examples of sets for which such a bound does hold, and examples for which such a bound does not hold. The classification of sets with respect to such bounds is currently an open question. In particular, what conditions are necessary and sufficient to ensure that $\dim_A(\mathcal{A}-\mathcal{A}) \leq K\dim_A(\mathcal{A})$ for some finite constant K?

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