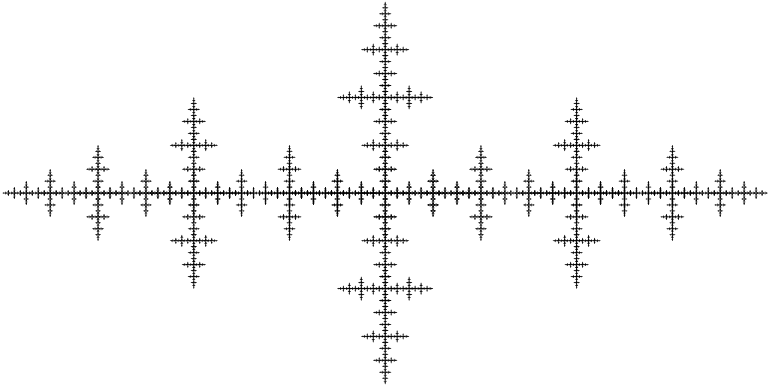
Abstract:

The Assouad dimension is a measure of the complexity of a fractal set similar to the box counting dimension, but with an additional scaling requirement. We generalize Moran's open set condition and introduce a notion called grid like which allows us to compute upper bounds and exact values for the Assouad dimension of certain fractal sets that arise as the attractors of self-similar iterated function systems. Then for an arbitrary fractal set $\mathcal{A}$, we explore the question of whether the Assouad dimension of the set of differences $\mathcal{A} - \mathcal{A}$ obeys any bound related to the Assouad dimension of $\mathcal{A}$. This question is of interest, as infinite dimensional dynamical systems with attractors possessing sets of differences of finite Assouad dimension allow embeddings into finite dimensional spaces without losing the original dynamics. We find that even in very simple, natural examples, such a bound does not generally hold. This result demonstrates how a natural phenomenon with a simple underlying structure can be difficult to measure.

Assoud Dimension and the Open Set Condition

- Jouni Luukkainen.
  Assouad dimension: Antifractal metrization, porous sets, and homogeneous measures.
  Journal of the Korean Mathematical Society, 35(1):23–76, 1998.
- Eric J. Olson and James C. Robinson.
  Almost bi-Lipschitz embeddings and almost homogeneous sets.
  Transactions of the American Mathematical Society, 362:145–168, 2010.
- K. J. Falconer.
  The Geometry of Fractal Sets.
  Cambridge University Press, New York, 1985.
- John M. Mackay.
  Assouad dimension of self-affine carpets.
  Conformal Geometry and Dynamics, 15:177–187, 2011.
- G. H. Hardy, E. M. Wright, D. R. Heath-Brown, and J.H. Silverman.
  An Introduction to the Theory of Numbers.
  Oxford University Press, Oxford, 2008.

Example



Question

Does the Moran open set condition imply that $\dim_f(\mathcal{A}) = \dim_A(\mathcal{A})$?

Question

Does there exist a bound of the form $\dim_A(\mathcal{A} - \mathcal{A}) \le K \dim_A(\mathcal{A})$?

Definition

An *iterated function system* is a collection $F = \{f_i\}_{i=1}^{L}$ of 2 or more continuous maps on $\mathbb{R}^D$ with the property that for each map f_i, there exists a constant $c_i \in (0,1)$ such that $\|f_i(\mathbf{x}) - f_i(\mathbf{y})\| \leq c_i \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$. The constant c_i is called the *contraction ratio* of f_i.

Theorem (Hutchinson)

If $F = \{f_i\}_{i=1}^{L}$ is an iterated function system, then there exists a unique, non-empty, compact set $\mathcal{A}$ such that

$$\mathcal{A} = \bigcup_{i=1}^{L} f_i(\mathcal{A}).$$

This set is called the attractor of F.

Hausdorff Dimension (Besicovich, Hausdorff)

The *Hausdorff dimension* of $\mathcal{A}$, denoted $\dim_H(\mathcal{A})$, is the infimal value of d such that

$$\liminf_{\rho \to 0} \left\{ \sum_{i=1}^{\infty} \rho_i^d \;\middle|\; \mathcal{A} \subseteq \bigcup_{i=1}^{\infty} \mathrm{B}_{\rho_i}(x_i) \text{ and } \rho_i < \rho \right\} = 0.$$

Fractal Dimension (Bouligand, Minkowski)

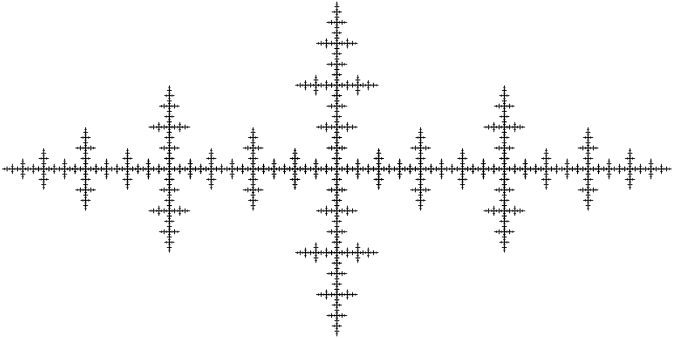
Let $\mathcal{N}_{\mathcal{A}}(\rho)$ denote the minimum number of ρ-balls centered in $\mathcal{A}$ required to cover $\mathcal{A}$. The *fractal dimension* of $\mathcal{A}$, denoted $\dim_f(\mathcal{A})$, is the infimal value of d for which there exists a constant K such that for any $0 < \rho < 1$,

$$\mathcal{N}_{\mathcal{A}}(\rho) \le K \left(1/\rho\right)^d.$$

Assouad Dimension (Assouad, Bouligand)

Let $\mathcal{N}_{\mathcal{A}}(r, \rho)$ denote the number ρ-balls centered in $\mathcal{A}$ required to cover any r-ball centered in $\mathcal{A}$. The *Assouad dimension* of $\mathcal{A}$, denoted $\dim_A(\mathcal{A})$, is the infimal value of d for which there exists a constant K such that for any $0 < \rho < r < 1$,

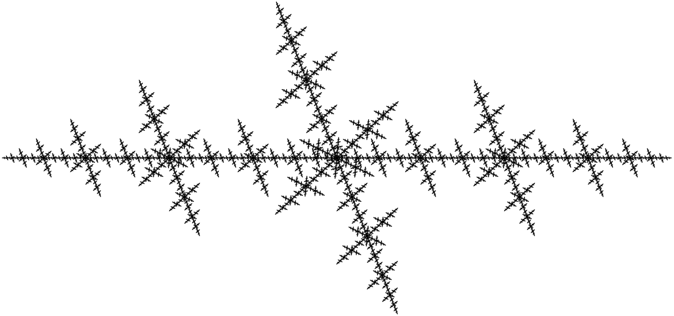
$$\mathcal{N}_{\mathcal{A}}(r, \rho) \le K \left(r/\rho\right)^d.$$



This set is the attractor $\mathcal{A}$ of the iterated function system $F = \{f_i\}_{i=1}^3$ with maps

$$f_1(\mathbf{x}) = \frac{1}{2}\mathbf{x} - \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad f_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad f_3(\mathbf{x}) = \frac{1}{2}R_\theta \mathbf{x},$$

where $\theta = \pi/2$. For this set, $\dim_f(\mathcal{A}) = \dim_A(\mathcal{A})$.



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where $\theta = 2\pi/(1+\sqrt{5})$. For this set, $\dim_f(\mathcal{A}) < \dim_A(\mathcal{A})$.

Definition

An iterated function system $F = \{f_i\}_{i=1}^{L}$ is said to satisfy the *Moran open set condition* if there exists a non-empty open set U such that $f_i(U) \subseteq U$ for each i, and $f_i(U) \cap f_j(U) = \emptyset$ whenever $i \neq j$.

Theorem (Hutchinson)

Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system of similarities with contraction ratio c_i corresponding to the map f_i for each i. If F satisfies the Moran open set condition, then

$$\dim_f(\mathcal{A}) = s$$

where s is the unique real number such that $\sum_{i=1}^{L} c_i^s = 1$. We call this value s the similarity dimension of $\mathcal{A}$, denoted $s = \dim_s(\mathcal{A})$.

- Does a similar result hold for the Assouad dimension?

- **(Luukkainen, 2008)**
  Is the Moran open set condition sufficient to ensure that $\dim_A(\mathcal{A}) = \dim_s(\mathcal{A})$?
- **(Mackay & Tyson, 2010)**
  Yes. The attractor of a self-similar iterated function system which satisfies the Moran open set condition is Ahlfors-regular, and the Hausdorff and Assouad dimensions of any Ahlfors-regular space coincide.
- **(Henderson, 2011)**
  An independent proof which generalizes the Moran open set condition and also gives upper bounds on the Assouad dimension for a class of sets that occur as the attractors of grid-like iterated function systems.

Theorem (Olson & Robinson, 2010)

Let $\mathcal{A}$ be a compact subset of a Hilbert space $\mathscr{H}$ such that $\mathcal{A}-\mathcal{A}$ is (α,β)-almost homogeneous with $\dim_A^{\alpha,\beta}(\mathcal{A}-\mathcal{A}) < d < D$. If

$$\gamma > \frac{2 + D(3+\alpha+\beta) + 2(\alpha+\beta)}{2(D-d)}$$

then a prevalent set of linear maps $f\colon \mathscr{H} \to \mathbb{R}^D$ are injective on $\mathcal{A}$ and, in particular, γ-almost bi-Lipschitz.

- It can be shown that $\dim_f(\mathcal{A}-\mathcal{A}) \leq 2\dim_f(\mathcal{A})$. A similar bound for the Assouad dimension of the set of differences is desirable.
- There exist abstract examples of sets with small Assouad dimension that possess sets of differences of large Assouad dimension.
- Self-similar iterated function systems which satisfy the Moran open set condition are extraordinarily structured. If $\mathcal{A}$ is the attractor of such a system, can bounds on $\dim_A(\mathcal{A}-\mathcal{A})$ be obtained in terms of $\dim_A(\mathcal{A})$?

Middle-λ Cantor Sets

Fix $\lambda \in (1/3, 1)$ and define $c = (1-\lambda)/2$. The middle-λ Cantor set $\mathcal{C}_\lambda$ is the attractor of the iterated function system $F_\lambda = \{f_1, f_2\}$ with maps on $\mathbb{R}$ given by

$$f_1(x) = cx, \qquad \text{and} \qquad f_2(x) = cx + (1-c).$$

Then

$$\dim_A(\mathcal{C}_\lambda) = \frac{\log(2)}{\log(\frac{1}{c})} \qquad \text{and} \qquad \dim_A(\mathcal{C}_\lambda - \mathcal{C}_\lambda) = \frac{\log(3)}{\log(\frac{1}{c})}.$$

Asymmetric Cantor Sets

Fix $c_1, c_2 \in (0,1)$ with $c_1 + c_2 < 1$. The asymmetric Cantor set $\mathcal{A}_{c_1,c_2}$ is the attractor of the iterated function system $F_{c_1,c_2} = \{f_1, f_2\}$ with maps given by

$$f_1(x) = c_1 x, \qquad \text{and} \qquad f_2(x) = c_2 x + (1-c_2).$$

For most choices of c_1 and c_2, $\dim_A(\mathcal{A}_{c_1,c_2} - \mathcal{A}_{c_1,c_2}) = 1$, even if $\dim_A(\mathcal{A}_{c_1,c_2})$ is arbitrarily small.

Definition

Consider the inequality

$$\left|\frac{p}{q} - \xi\right| < \frac{C}{q^{2+\varepsilon}}.$$

We say that an irrational number ξ is *well approximable by rationals* if for every $\varepsilon > 0$, there are infinitely many q such that this inequality is satisfied. Otherwise, we say that ξ is *badly approximable by rationals*.

Theorem (Henderson)

If $\frac{\log(c_1)}{\log(c_2)}$ *is badly approximable by rationals, then* $\dim_A(\mathcal{A}_{c_1,c_2} - \mathcal{A}_{c_1,c_2}) = 1$.

Theorem (Henderson)

Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system of similarities in $\mathbb{R}^D$ with Moran open set U. Let $\mathcal{A}$ be the invariant set of F, and suppose that the contraction ratio of f_i is $c \in (0,1)$ for each i. Then $\dim_A(\mathcal{A}) = \log(L)/\log(1/c)$.

Definition

The *Assouad dimension* of $\mathcal{A}$ is the infimal value of a for which there exists a constant K such that for any $0 < \rho < r < 1$,

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \leq K \left(\frac{r}{\rho}\right)^d.$$

Fact

In this setting, $\log(L)/\log(1/c) = \dim_f(\mathcal{A}) \leq \dim_A(\mathcal{A})$, thus it is sufficient to show that $\dim_A(\mathcal{A}) \leq \log(L)/\log(1/c)$.

Miscellaneous Ingredients

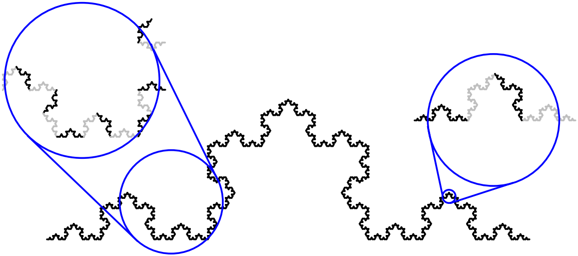
- The length of a finite sequence α is denoted $\ell(\alpha)$
- $\mathcal{A} \subseteq \overline{U} \subseteq \mathbb{R}^D \qquad \mathcal{A} = \bigcup\{f^\beta(\mathcal{A}) \mid \ell(\beta) = n\}$
- $\mathcal{N}_\mathcal{A}(r, \rho)$ is the number of ρ-balls in $\mathcal{A}$ required to cover an r-ball in $\mathcal{A}$
- $\delta = \operatorname{diam}(U) \qquad \nu = \lambda^D(U) \qquad \Omega_D = \lambda^D(\mathrm{B}_1(\mathbf{0}))$
- If $\ell(\alpha) = m$, then
  - $\operatorname{diam}(f^\alpha(\mathcal{A})) = c^m \operatorname{diam}(\mathcal{A})$
  - $\lambda^D(f^\alpha(U)) = c^{mD} \lambda^D(U)$

Lemma

In this setting, if $\ell(\alpha) = \ell(\tilde{\alpha})$ and $\alpha \neq \tilde{\alpha}$, then $f^\alpha(U) \cap f^{\tilde{\alpha}}(U) = \emptyset$.

The End

- **Applied Mathematics**
  Embeddings of Dynamical Systems
- **Experimental Data**
  Potential Complexity of Measured Data
- **Number Theory**
  New Language for Describing Badly Approximable Numbers



Definition

Let $F = \{f_i\}_{i=1}^{L}$ be an iterated function system with attractor $\mathcal{A}$ in $\mathbb{R}^D$. F is said to be *grid like* if there exists $N \in \mathbb{N}$ such that for every $r > 0$ and any $p \in \mathbb{R}^D$, there is a set $A \subseteq \mathcal{S}_L$ such that

1. $\operatorname{card} A \leq N$,
2. $\operatorname{diam}(f^\alpha(\mathcal{A})) < r$ for each $\alpha \in A$, and
3. $\mathcal{A} \cap \mathrm{B}_r(p) \subseteq \bigcup_{\alpha \in A} f^\alpha(\mathcal{A})$.

- **Grid Like Systems and the Open Set Condtion**
  It is sometimes possible to compute the Assouad dimension of the attractor of a grid like system by constructing another system that has the same attractor and which satisfies the open set condition. Under what circumstances can this be done?
- **Self-Affine Sets**
  Mackay & Tyson and Fraser have recently computed the Assouad dimension of certain two-dimensional self-affine sets. Their methods use properties of projections and cross-sections of the self-affine sets. Can their techniques be modified to obtain results if the projections and/or cross-sections satisfy the grid like condition?
- **Self-Similar Iterated Function Systems in Hilbert Space**
  Suppose that $F = \{f_i\}_{i=1}^{\infty}$ is an iterated function system on $\mathscr{H}$. Under what circumstances does F possess a compact invariant set?
- **Classification of Sets of Differences**
  There are examples and counterexamples of sets $\mathcal{A}$ that satisfy bounds of the form $\dim_A(\mathcal{A} - \mathcal{A}) \leq 2 \dim_A(\mathcal{A})$. Can a general classification of sets be found relative to such bounds?