The Complex Dimensions of Self-Similar Subsets of p-adic Product Spaces

Alexander M. Henderson

University of California Riverside
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- Homogeneous measures
- The distance zeta function
- p-adic spaces
- Iterated function systems on $\mathbb{Q}_p^Q$

Results & Examples

- Self-similar sets
- 3-adic Cantor dust
- Fibonacci attractors
- A McMullen carpet analog

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Definitions & Notation

Definitions & Notation: Homogeneous measures

Let (X, d, μ) be a complete, separable metric measure space such that

$$0 < \mu(B(x,r)) < \infty$$

for all $x \in X$ and $r > 0$. Let $A \subseteq X$.

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Definition

We say that μ is ***q-homogeneous*** on A if there is some constant $M > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(\xi,\rho))} \leq M \left(\frac{r}{\rho}\right)^q$$

for all $0 < \rho < r \leq \operatorname{diam}(A)$, all $x \in A$, and all $\xi \in B(x,r)$.

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for all $0 < \rho < r \leq \operatorname{diam}(A)$, all $x \in A$, and all $\xi \in B(x, r)$. The ***measure theoretic Assouad dimension*** of A is

$$\dim_{\mathrm{As}}(A) := \inf \{q \geq 0 \mid \mu \text{ is } q\text{-homogeneous on } A\}.$$

Definitions & Notation: The distance zeta function

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Suppose that $\dim_{\mathrm{As}}(X) = Q$ and that A is a bounded subset of X. For $\delta > 0$, define

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Definition

Suppose that $\zeta_A(s)$ can be meromorphically extended to a (strictly) larger domain. Then the ***complex dimensions*** of A, denoted $\mathscr{P}(A)$, are the poles of this extension. That is

$$\mathscr{P}(A) := \{\omega \in \mathbb{C} \,|\, \omega \text{ is a pole of } \zeta_A(s)\}.$$

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Let p be a fixed prime number.

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$$\mathbb{Z}_p := B_{\leq}(0,1) = \{x \in \mathbb{Q}_p \,|\, |x|_p \leq 1\}.$$

$\mathbb{Q}_p$ is equipped with the Haar measure μ such that $\mu(\mathbb{Z}_p) = 1$.

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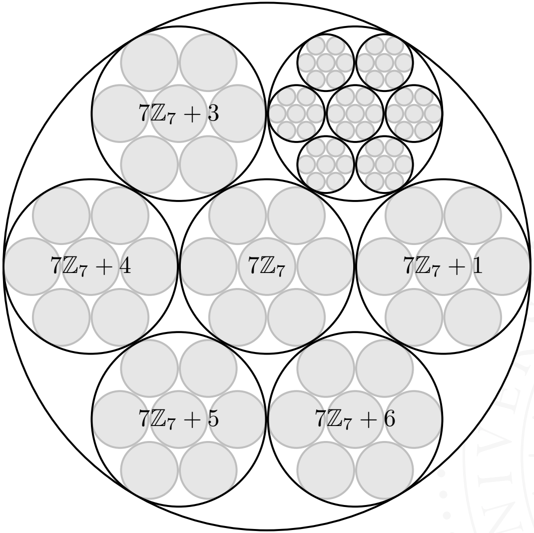

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Let $Q \in \mathbb{N}$ and $\alpha \in [1, \infty)$.

Notation

On the product space $\mathbb{Q}_p^Q$, define the equivalent metrics

$$d^\alpha(\boldsymbol{x}, \boldsymbol{y}) := \left(\sum_{i=1}^{Q} |x_i - y_i|_p^\alpha \right)^{1/\alpha},$$

and

$$d^\infty(\boldsymbol{x}, \boldsymbol{y}) := \max \left\{ |x_i - y_i|_p \,\middle|\, 1 \leq i \leq Q \right\}.$$

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Lemma

For any $Q \in \mathbb{N}$ and any $\alpha \in [1, \infty]$, the product space $(\mathbb{Q}_p^Q, d^\alpha, \mu)$ satisfies

$$\dim_{\mathrm{As}}(\mathbb{Q}_p^Q) = Q,$$

where μ is the natural product measure.

Definitions & Notation: Iterated function systems on $\mathbb{Q}_p^Q$

Definition

A ***self-similar iterated function system*** (SSIFS) on $\mathbb{Q}_p^Q$ is a finite collection of maps $\{\varphi_j\}_{j\in\mathscr{J}}$, each of which is of the form

$$\varphi_j(x) = p^{k_j}x + b_j,$$

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$$\Phi(E) := \bigcup_{j\in\mathscr{J}} \varphi_j(E).$$

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$$\Phi(E) := \bigcup_{j\in\mathscr{J}} \varphi_j(E).$$

Theorem

Let Φ be as above. Then there is a unique, nonempty, compact set $\mathscr{A} \subseteq \mathbb{Q}_p^Q$ such that

$$\Phi(\mathscr{A}) = \mathscr{A}.$$

*We call $\mathscr{A}$ the **attractor** of the SSIFS.*

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Let $\{\varphi_j\}_{j\in\mathscr{J}}$ be an SSIFS.

Notation

Let $\mathscr{J}^*$ denote the set of all finite sequences (or "words") with entries in $\mathscr{J}$. For each

$$J = (j_1, j_2, \ldots, j_n) \in \mathscr{J},$$

define

$$\varphi_J = \varphi_{j_n} \circ \varphi_{j_{n-1}} \circ \cdots \circ \varphi_1.$$

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Let $\omega = (\ ) \in \mathscr{J}^*$ denote the "empty word." We adopt the convention that φ_ω is the identity map, i.e.

$$\varphi_\omega(x) = x.$$

Results & Examples

Results & Examples: Self-similar sets

Theorem

Fix $\alpha \in [1, \infty]$ *and let* $\mathscr{A}$ *be the attractor of the SSIFS* $\{\varphi_j\}_{j \in \mathscr{J}}$ *on* $\mathbb{Q}_p^Q$.

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$$\zeta_{\mathscr{A}}(s) = \zeta_{\mathscr{A}, \Omega_\iota}(s) \sum_{n=0}^{\infty} C_n p^{-ns},$$

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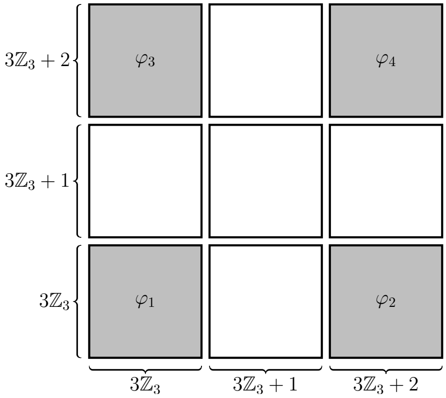
and C_n counts the number of maps of the form φ_J for some $J \in \mathscr{J}^*$ with contraction ratio p^{-n}.

Results & Examples: 3-adic Cantor dust



Example

Let $\{\varphi_j\}_{j=1}^4$ *be the SSIFS on* $\mathbb{Q}_3^2$ *that maps* $\mathbb{Z}_3^2$ *into the four rectangles shown to the left. Let* $\mathscr{C}^2$ *denote the attractor of this SSIFS.*

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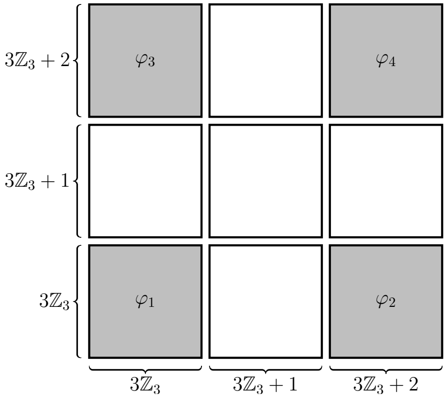

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We may also regard $\mathscr{C}^2$ as the Cartesian product of two copies of a 3-adic Cantor set. In either case, $\mathscr{C}^2$ is an analog of the ternary Cantor dust in $\mathbb{R}^2$.

Results & Examples: 3-adic Cantor dust

Example (con't)

With respect to d^∞,

$$\zeta_{\mathscr{C}^2,\Omega_\iota}(s) = \int_{\mathbb{Z}_3^2 \setminus \Phi(\mathbb{Z}_3^2)} d^\infty(x,\mathscr{C}^2)^{s-2}\,\mathrm{d}\mu(x) = \mu\left(\mathbb{Z}_3^2 \setminus \Phi(\mathbb{Z}_3^2)\right) = \frac{5}{9}.$$

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Next, observe that

$$C_n := \#\left\{J \in \mathscr{J}^* \,|\, \varphi_J(x) = 3^n x + b_J\right\} = 4^n.$$

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Therefore

$$\mathscr{P}(\mathscr{C}^2) = \frac{\log(4)}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}.$$

Results & Examples: Fibonacci attractors

Example

Fix a prime p and define maps on $\mathbb{Q}_p$ by

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Hence

$$\sum_{n=0}^{\infty} C_n p^{-ns} = \frac{\sqrt{5} p^{2s}}{(p^s - \phi)(p^s - \psi)}.$$

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And so

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Therefore

$$\mathscr{P}(\mathscr{F}) = \left(\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}\right) \cup \left(-\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{(2\pi+1)\mathbb{Z}}{\log(p)}\right).$$

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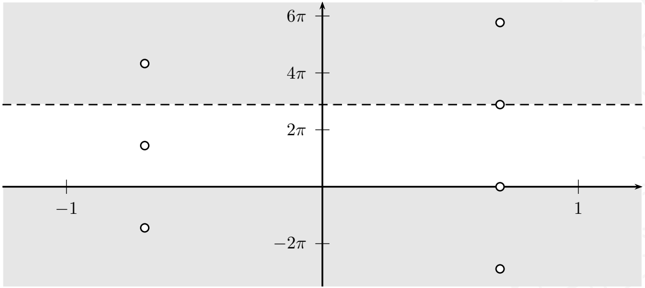
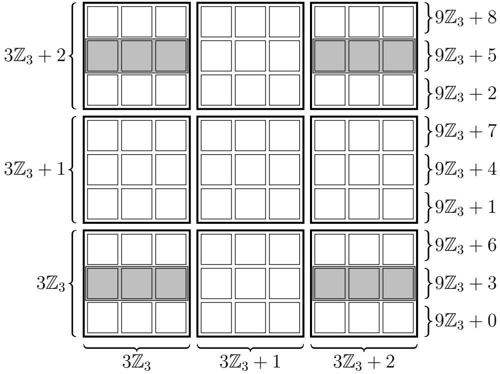
$$\mathscr{P}(\mathscr{F}) = \left(\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}\right) \cup \left(-\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{(2\pi+1)\mathbb{Z}}{\log(p)}\right).$$



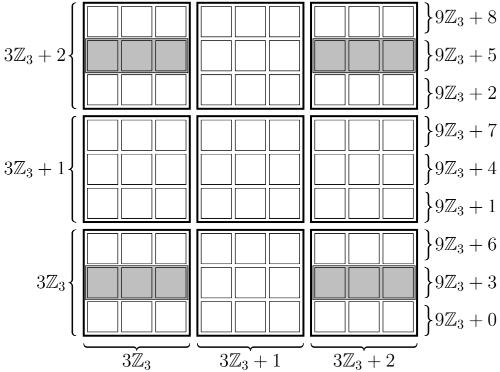
Results & Examples: A McMullen carpet analog



Example

Let $\mathscr{A}$ denote the attractor of the IFS shown to the left.

Results & Examples: A McMullen carpet analog



Example

Let $\mathscr{A}$ denote the attractor of the IFS shown to the left. With respect to d^∞,

$$\mathscr{P}(\mathscr{A}) = \left(\frac{3\log(2)}{2\log(3)} + \mathbb{i}\frac{\pi\mathbb{Z}}{\log(3)}\right) \cup \left(\frac{\log(4)}{\log(3)} - 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}\right).$$

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[1] Michel L. Lapidus and Hùng Lũ', *Nonarchimedean cantor set and string*, J. Fixed Point Theory and Appl. **3** (2008), no. 1, 181–190.

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[3] Curt McMullen, *The Hausdorff dimension of general Sierpński carpets*, Nagoya Mathematical J. **96** (1984), 1–9.